Dynamic Stackelberg Problems

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1 Contents

This notebook formulates and computes a plan that a **Stackelberg leader** uses to manipulate forward-looking decisions of a **Stackelberg follower** that depend on continuation sequences of decisions made once and for all by the Stackelberg leader at time 0.

To facilitate computation and interpretation, we formulate things in a context that allows us to apply linear optimal dynamic programming.

From the beginning we carry along a linear-quadratic model of duopoly in which firms face adjustment costs that make them want to forecast actions of other firms that influence future prices.

2 Duopoly

Time is discrete and is indexed by $t = 0, 1, \ldots$.

Two firms produce a single good whose demand is governed by the linear inverse demand curve

$$p_t = a_0 - a_1(q_{1t} + q_{2t})$$

where q_{it} is output of firm i at time t and a_0 and a_1 are both positive.

q_{10}, q_{20} are given numbers that serve as initial conditions at time 0.

By incurring a cost of change

$$\gamma v_{it}^2$$

where $\gamma > 0$, firm i can change its output according to

$$q_{it+1} = q_{it} + v_{it}$$

Firm i's profits at time t equal

$$\pi_{it} = p_t q_{it} - \gamma v_{it}^2$$

Firm i wants to maximize the present value of its profits

$$\sum_{t=0}^{\infty} \beta^t \pi_{it}$$

where $\beta \in (0,1)$ is a time discount factor.

2.1 Stackelberg Leader and Follower

Each firm $i = 1,2$ chooses a sequence $\vec{q}_i \equiv \{q_{it+1}\}_{t=0}^{\infty}$ once and for all at time 0.

We let firm 2 be a **Stackelberg leader** and firm 1 be a **Stackelberg follower**.

The leader firm 2 goes first and chooses $\{q_{2t+1}\}_{t=0}^{\infty}$ once and for all at time 0.

Knowing that firm 2 has chosen $\{q_{2t+1}\}_{t=0}^{\infty}$, the follower firm 1 goes second and chooses $\{q_{1t+1}\}_{t=0}^{\infty}$ once and for all at time 0.

In choosing $\vec{q}_2$, firm 2 takes into account that firm 1 will base its choice of $\vec{q}_1$ on firm 2's choice of $\vec{q}_2$.

2.2 Abstract Statement of the Leader's and Follower's Problems

We can express firm 1's problem as

$$\max_{\vec{q}_1} \Pi_1(\vec{q}_1; \vec{q}_2)$$

where the appearance behind the semi-colon indicates that $\vec{q}_2$ is given.

Firm 1's problem induces a best response mapping

$$\vec{q}_1 = B(\vec{q}_2)$$

(Here B maps a sequence into a sequence)

The Stackelberg leader's problem is

$$\max_{\vec{q}_2} \Pi_2(B(\vec{q}_2), \vec{q}_2)$$

whose maximizer is a sequence $\vec{q}_2$ that depends on the initial conditions q_{10}, q_{20} and the parameters of the model a_0, a_1, γ.

This formulation captures key features of the model

- Both firms make once-and-for-all choices at time 0.
- This is true even though both firms are choosing sequences of quantities that are indexed by **time**.
- The Stackelberg leader chooses first **within time** 0, knowing that the Stackelberg follower will choose second **within time** 0.

While our abstract formulation reveals the timing protocol and equilibrium concept well, it obscures details that must be addressed when we want to compute and interpret a Stackelberg plan and the follower's best response to it.

To gain insights about these things, we study them in more detail.

2.3 Firms' Problems

Firm 1 acts as if firm 2's sequence $\{q_{2t+1}\}_{t=0}^{\infty}$ is given and beyond its control.

Firm 2 knows that firm 1 chooses second and takes this into account in choosing $\{q_{2t+1}\}_{t=0}^{\infty}$.

In the spirit of *working backwards*, we study firm 1's problem first, taking $\{q_{2t+1}\}_{t=0}^{\infty}$ as given.

We can formulate firm 1's optimum problem in terms of the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \{a_0 q_{1t} - a_1 q_{1t}^2 - a_1 q_{1t} q_{2t} - \gamma v_{1t}^2 + \lambda_t [q_{1t} + v_{1t} - q_{1t+1}]\}$$

Firm 1 seeks a maximum with respect to $\{q_{1t+1}, v_{1t}\}_{t=0}^{\infty}$ and a minimum with respect to $\{\lambda_t\}_{t=0}^{\infty}$.

We approach this problem using methods described in Ljungqvist and Sargent RMT5 chapter 2, appendix A and Macroeconomic Theory, 2nd edition, chapter IX.

First-order conditions for this problem are

$$\begin{aligned}
\frac{\partial L}{\partial q_{1t}} &= a_0 - 2a_1 q_{1t} - a_1 q_{2t} + \lambda_t - \beta^{-1}\lambda_{t-1} = 0, \quad t \geq 1 \\
\frac{\partial L}{\partial v_{1t}} &= -2\gamma v_{1t} + \lambda_t = 0, \quad t \geq 0
\end{aligned}$$

These first-order conditions and the constraint $q_{1t+1} = q_{1t} + v_{1t}$ can be rearranged to take the form

$$\begin{aligned}
v_{1t} &= \beta v_{1t+1} + \frac{\beta a_0}{2\gamma} - \frac{\beta a_1}{\gamma} q_{1t+1} - \frac{\beta a_1}{2\gamma} q_{2t+1} \\
q_{t+1} &= q_{1t} + v_{1t}
\end{aligned}$$

We can substitute the second equation into the first equation to obtain

$$(q_{1t+1} - q_{1t}) = \beta(q_{1t+2} - q_{1t+1}) + c_0 - c_1 q_{1t+1} - c_2 q_{2t+1}$$

where $c_0 = \frac{\beta a_0}{2\gamma}, c_1 = \frac{\beta a_1}{\gamma}, c_2 = \frac{\beta a_1}{2\gamma}$.

This equation can in turn be rearranged to become the second-order difference equation

$$q_{1t} + (1 + \beta + c_1) q_{1t+1} - \beta q_{1t+2} = c_0 - c_2 q_{2t+1} \tag{1}$$

Equation (1) is a second-order difference equation in the sequence $\vec{q}_1$ whose solution we want.

It satisfies **two boundary conditions:**

- an initial condition that $q_{1,0}$, which is given
- a terminal condition requiring that $\lim_{T \to +\infty} \beta^T q_{1t}^2 < +\infty$

Using the lag operators described in chapter IX of *Macroeconomic Theory, Second edition (1987)*, difference equation (1) can be written as

$$\beta(1 - \frac{1 + \beta + c_1}{\beta} L + \beta^{-1} L^2) q_{1t+2} = -c_0 + c_2 q_{2t+1}$$

The polynomial in the lag operator on the left side can be **factored** as

$$(1 - \frac{1 + \beta + c_1}{\beta} L + \beta^{-1} L^2) = (1 - \delta_1 L)(1 - \delta_2 L) \tag{2}$$

where $0 < \delta_1 < 1 < \frac{1}{\sqrt{\beta}} < \delta_2$.

Because $\delta_2 > \frac{1}{\sqrt{\beta}}$ the operator $(1 - \delta_2 L)$ contributes an **unstable** component if solved **backwards** but a **stable** component if solved **forwards**.

Mechanically, write

$$(1 - \delta_2 L) = -\delta_2 L (1 - \delta_2^{-1} L^{-1})$$

and compute the following inverse operator

$$\left[-\delta_2 L (1 - \delta_2^{-1} L^{-1})\right]^{-1} = -\delta_2 (1 - {\delta_2}^{-1})^{-1} L^{-1}$$

Operating on both sides of equation (2) with β^{-1} times this inverse operator gives the follower's decision rule for setting q_{1t+1} in the **feedback-feedforward** form

$$q_{1t+1} = \delta_1 q_{1t} - c_0 \delta_2^{-1} \beta^{-1} \frac{1}{1 - \delta_2^{-1}} + c_2 \delta_2^{-1} \beta^{-1} \sum_{j=0}^{\infty} \delta_2^j q_{2t+j+1}, \quad t \geq 0 \tag{3}$$

The problem of the Stackelberg leader firm 2 is to choose the sequence $\{q_{2t+1}\}_{t=0}^{\infty}$ to maximize its discounted profits

$$\sum_{t=0}^{\infty} \beta^t \{(a_0 - a_1(q_{1t} + q_{2t})) q_{2t} - \gamma (q_{2t+1} - q_{2t})^2\}$$

subject to the sequence of constraints (3) for $t \geq 0$.

We can put a sequence $\{\theta_t\}_{t=0}^{\infty}$ of Lagrange multipliers on the sequence of equations (3) and formulate the following Lagrangian for the Stackelberg leader firm 2's problem

$$
\begin{aligned}
\tilde{L} = & \sum_{t=0}^{\infty} \beta^t \{(a_0 - a_1(q_{1t} + q_{2t}))q_{2t} - \gamma(q_{2t+1} - q_{2t})^2\} \\
& + \sum_{t=0}^{\infty} \beta^t \theta_t \{\delta_1 q_{1t} - c_0 \delta_2^{-1} \beta^{-1} \frac{1}{1 - \delta_2^{-1}} + c_2 \delta_2^{-1} \beta^{-1} \sum_{j=0}^{\infty} \delta_2^{-j} q_{2t+j+1} - q_{1t+1}
\end{aligned} \tag{4}
$$

subject to initial conditions for q_{1t}, q_{2t} at $t = 0$.

Comments: We have formulated the Stackelberg problem in a space of sequences.

The max-min problem associated with Lagrangian (4) is unpleasant because the time t component of firm 1's payoff function depends on the entire future of its choices of $\{q_{1t+j}\}_{j=0}^{\infty}$.

This renders a direct attack on the problem cumbersome.

Therefore, below, we will formulate the Stackelberg leader's problem recursively.

We'll put our little duopoly model into a broader class of models with the same conceptual structure.

3 The Stackelberg Problem

We formulate a class of linear-quadratic Stackelberg leader-follower problems of which our duopoly model is an instance.

We use the optimal linear regulator (a.k.a. the linear-quadratic dynamic programming problem described in LQ Dynamic Programming problems) to represent a Stackelberg leader's problem recursively.

Let z_t be an $n_z \times 1$ vector of **natural state variables**.

Let x_t be an $n_x \times 1$ vector of endogenous forward-looking variables that are physically free to jump at t.

In our duopoly example $x_t = v_{1t}$, the time t decision of the Stackelberg **follower**.

Let u_t be a vector of decisions chosen by the Stackelberg leader at t.

The z_t vector is inherited physically from the past.

But x_t is a decision made by the Stackelberg follower at time t that is the follower's best response to the choice of an entire sequence of decisions made by the Stackelberg leader at time $t = 0$.

Let

$$
y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}
$$

Represent the Stackelberg leader's one-period loss function as

$$
r(y, u) = y' R y + u' Q u
$$

Subject to an initial condition for z_0, but not for x_0, the Stackelberg leader wants to maximize

$$-\sum_{t=0}^{\infty} \beta^t r(y_t, u_t) \tag{5}$$

The Stackelberg leader faces the model

$$\begin{bmatrix} I & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \hat{B} u_t \tag{6}$$

We assume that the matrix $\begin{bmatrix} I & 0 \\ G_{21} & G_{22} \end{bmatrix}$ on the left side of equation (6) is invertible, so that we can multiply both sides by its inverse to obtain

$$\begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + B u_t \tag{7}$$

or

$$y_{t+1} = A y_t + B u_t \tag{8}$$

3.1 Interpretation of the Second Block of Equations

The Stackelberg follower's best response mapping is summarized by the second block of equations of (7).

In particular, these equations are the first-order conditions of the Stackelberg follower's optimization problem (i.e., its Euler equations).

These Euler equations summarize the forward-looking aspect of the follower's behavior and express how its time t decision depends on the leader's actions at times $s \geq t$.

When combined with a stability condition to be imposed below, the Euler equations summarize the follower's best response to the sequence of actions by the leader.

The Stackelberg leader maximizes (5) by choosing sequences $\{u_t, x_t, z_{t+1}\}_{t=0}^{\infty}$ subject to (8) and an initial condition for z_0.

Note that we have an initial condition for z_0 but not for x_0.

x_0 is among the variables to be chosen at time 0 by the Stackelberg leader.

The Stackelberg leader uses its understanding of the responses restricted by (8) to manipulate the follower's decisions.

3.2 More Mechanical Details

For any vector a_t, define $\vec{a}_t = [a_t, a_{t+1} \ldots]$.

Define a feasible set of $(\vec{y}_1, \vec{u}_0)$ sequences

$$\Omega(y_0) = \{(\vec{y}_1, \vec{u}_0) : y_{t+1} = A y_t + B u_t, \forall t \geq 0\}$$

Please remember that the follower's Euler equation is embedded in the system of dynamic equations $y_{t+1} = Ay_t + Bu_t$.

Note that in the definition of $\Omega(y_0)$, y_0 is taken as given.

Although it is taken as given in $\Omega(y_0)$, eventually, the x_0 component of y_0 will be chosen by the Stackelberg leader.

3.3 Two Subproblems

Once again we use backward induction.

We express the Stackelberg problem in terms of **two subproblems**.

Subproblem 1 is solved by a **continuation Stackelberg leader** at each date $t \geq 0$.

Subproblem 2 is solved the **Stackelberg leader** at $t = 0$.

The two subproblems are designed

- to respect the protocol in which the follower chooses $\vec{q}_1$ after seeing $\vec{q}_2$ chosen by the leader
- to make the leader choose $\vec{q}_2$ while respecting that $\vec{q}_1$ will be the follower's best response to $\vec{q}_2$
- to represent the leader's problem recursively by artfully choosing the state variables confronting and the control variables available to the leader

3.3.1 Subproblem 1

$$v(y_0) = \max_{(\vec{y}_1, \vec{u}_0) \in \Omega(y_0)} - \sum_{t=0}^{\infty} \beta^t r(y_t, u_t)$$

3.3.2 Subproblem 2

$$w(z_0) = \max_{x_0} v(y_0)$$

Subproblem 1 takes the vector of forward-looking variables x_0 as given.

Subproblem 2 optimizes over x_0.

The value function $w(z_0)$ tells the value of the Stackelberg plan as a function of the vector of natural state variables at time 0, z_0.

3.4 Two Bellman Equations

We now describe Bellman equations for $v(y)$ and $w(z_0)$.

3.4.1 Subproblem 1

The value function $v(y)$ in subproblem 1 satisfies the Bellman equation

$$v(y) = \max_{u,y^*} \{-r(y,u) + \beta v(y^*)\} \tag{9}$$

where the maximization is subject to

$$y^* = Ay + Bu$$

and y^* denotes next period's value.

Substituting $v(y) = -y'Py$ into Bellman equation (9) gives

$$-y'Py = \max_{u,y^*} \{-y'Ry - u'Qu - \beta y^{*\prime} P y^*\}$$

which as in lecture linear regulator gives rise to the algebraic matrix Riccati equation

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1} B'PA$$

and the optimal decision rule coefficient vector

$$F = \beta(Q + \beta B'PB)^{-1} B'PA$$

where the optimal decision rule is

$$u_t = -F y_t$$

3.4.2 Subproblem 2

We find an optimal x_0 by equating to zero the gradient of $v(y_0)$ with respect to x_0:

$$-2P_{21} z_0 - 2P_{22} x_0 = 0,$$

which implies that

$$x_0 = -P_{22}^{-1} P_{21} z_0$$

4 Stackelberg Plan

Now let's map our duopoly model into the above setup.

We we'll formulate a state space system

$$y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$$

where in this instance $x_t = v_{1t}$, the time t decision of the follower firm 1.

4.1 Calculations to Prepare Duopoly Model

Now we'll proceed to cast our duopoly model within the framework of the more general linear-quadratic structure described above.

That will allow us to compute a Stackelberg plan simply by enlisting a Riccati equation to solve a linear-quadratic dynamic program.

As emphasized above, firm 1 acts as if firm 2's decisions $\{q_{2t+1}, v_{2t}\}_{t=0}^{\infty}$ are given and beyond its control.

4.2 Firm 1's Problem

We again formulate firm 1's optimum problem in terms of the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \{a_0 q_{1t} - a_1 q_{1t}^2 - a_1 q_{1t} q_{2t} - \gamma v_{1t}^2 + \lambda_t [q_{1t} + v_{1t} - q_{1t+1}]\}$$

Firm 1 seeks a maximum with respect to $\{q_{1t+1}, v_{1t}\}_{t=0}^{\infty}$ and a minimum with respect to $\{\lambda_t\}_{t=0}^{\infty}$.

First-order conditions for this problem are

$$\begin{aligned}
\frac{\partial L}{\partial q_{1t}} &= a_0 - 2a_1 q_{1t} - a_1 q_{2t} + \lambda_t - \beta^{-1}\lambda_{t-1} = 0, \quad t \geq 1 \\
\frac{\partial L}{\partial v_{1t}} &= -2\gamma v_{1t} + \lambda_t = 0, \quad t \geq 0
\end{aligned}$$

These first-order order conditions and the constraint $q_{1t+1} = q_{1t} + v_{1t}$ can be rearranged to take the form

$$\begin{aligned}
v_{1t} &= \beta v_{1t+1} + \frac{\beta a_0}{2\gamma} - \frac{\beta a_1}{\gamma} q_{1t+1} - \frac{\beta a_1}{2\gamma} q_{2t+1} \\
q_{t+1} &= q_{1t} + v_{1t}
\end{aligned}$$

We use these two equations as components of the following linear system that confronts a Stackelberg continuation leader at time t

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta a_0}{2\gamma} & -\frac{\beta a_1}{2\gamma} & -\frac{\beta a_1}{\gamma} & \beta \end{bmatrix} \begin{bmatrix} 1 \\ q_{2t+1} \\ q_{1t+1} \\ v_{1t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ q_{2t} \\ q_{1t} \\ v_{1t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} v_{2t}$$

Time t revenues of firm 2 are $\pi_{2t} = a_0 q_{2t} - a_1 q_{2t}^2 - a_1 q_{1t} q_{2t}$ which evidently equal

$$z_t' R_1 z_t \equiv \begin{bmatrix} 1 \\ q_{2t} \\ q_{1t} \end{bmatrix}' \begin{bmatrix} 0 & \frac{a_0}{2} & 0 \\ \frac{a_0}{2} & -a_1 & -\frac{a_1}{2} \\ 0 & -\frac{a_1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_{2t} \\ q_{1t} \end{bmatrix}$$

If we set $Q = \gamma$, then firm 2's period t profits can then be written

$$y_t' R y_t - Q v_{2t}^2$$

where

$$y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$$

with $x_t = v_{1t}$ and

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll report results of implementing this code soon.

But first we want to represent the Stackelberg leader's optimal choices recursively.

It is important to do this for several reasons:

- properly to interpret a representation of the Stackelberg leaders's choice as a sequence of history-dependent functions
- to formulate a recursive version of the follower's choice problem

First let's get a recursive representation of the Stackelberg leader's choice of $\vec{q}_2$ for our duopoly model.

5 Recursive Representation of Stackelberg Plan

In order to attain an appropriate representation of the Stackelberg leader's history-dependent plan, we will employ what amounts to a version of the **Big K, little k** device often used in macroeconomics by distinguishing z_t, which depends partly on decisions x_t of the followers, from another vector $\check{z}_t$, which does not.

We will use $\check{z}_t$ and its history $\check{z}^t = [\check{z}_t, \check{z}_{t-1}, \dots, \check{z}_0]$ to describe the sequence of the Stackelberg leader's decisions that the Stackelberg follower takes as given.

Thus, we let $\check{y}_t' = \begin{bmatrix} \check{z}_t' & \check{x}_t' \end{bmatrix}$ with initial condition $\check{z}_0 = z_0$ given.

That we distinguish $\check{z}_t$ from z_t is part and parcel of the **Big K, little k** device in this instance.

We have demonstrated that a Stackelberg plan for $\{u_t\}_{t=0}^\infty$ has a recursive representation

$$\begin{aligned} \check{x}_0 &= -P_{22}^{-1} P_{21} z_0 \\ u_t &= -F \check{y}_t, \quad t \geq 0 \\ \check{y}_{t+1} &= (A - BF) \check{y}_t, \quad t \geq 0 \end{aligned}$$

From this representation we can deduce the sequence of functions $\sigma = \{\sigma_t(\check{z}^t)\}_{t=0}^\infty$ that comprise a Stackelberg plan.

For convenience, let $\check{A} \equiv A - BF$ and partition $\check{A}$ conformably to the partition $y_t = \begin{bmatrix} \check{z}_t \\ \check{x}_t \end{bmatrix}$ as

$$\begin{bmatrix} \check{A}_{11} & \check{A}_{12} \\ \check{A}_{21} & \check{A}_{22} \end{bmatrix}$$

Let $H_0^0 \equiv -P_{22}^{-1} P_{21}$ so that $\check{x}_0 = H_0^0 \check{z}_0$.

Then iterations on $\check{y}_{t+1} = \check{A}\check{y}_t$ starting from initial condition $\check{y}_0 = \begin{bmatrix} \check{z}_0 \\ H_0^0 \check{z}_0 \end{bmatrix}$ imply that for $t \geq 1$

$$x_t = \sum_{j=1}^{t} H_j^t \check{z}_{t-j}$$

where

$$\begin{aligned} H_1^t &= \check{A}_{21} \\ H_2^t &= \check{A}_{22}\check{A}_{21} \\ \vdots \quad & \quad \vdots \\ H_{t-1}^t &= \check{A}_{22}^{t-2}\check{A}_{21} \\ H_t^t &= \check{A}_{22}^{t-1}(\check{A}_{21} + \check{A}_{22}H_0^0) \end{aligned}$$

An optimal decision rule for the Stackelberg's choice of u_t is

$$u_t = -F\check{y}_t \equiv -\begin{bmatrix} F_z & F_x \end{bmatrix} \begin{bmatrix} \check{z}_t \\ x_t \end{bmatrix}$$

or

$$u_t = -F_z\check{z}_t - F_x \sum_{j=1}^{t} H_j^t z_{t-j} = \sigma_t(\check{z}^t) \tag{10}$$

Representation (10) confirms that whenever $F_x \neq 0$, the typical situation, the time t component σ_t of a Stackelberg plan is **history dependent**, meaning that the Stackelberg leader's choice u_t depends not just on $\check{z}_t$ but on components of $\check{z}^{t-1}$.

5.1 Comments and Interpretations

After all, at the end of the day, it will turn out that because we set $\check{z}_0 = z_0$, it will be true that $z_t = \check{z}_t$ for all $t \geq 0$.

Then why did we distinguish $\check{z}_t$ from z_t?

The answer is that if we want to present to the Stackelberg **follower** a history-dependent representation of the Stackelberg **leader's** sequence $\vec{q}_2$, we must use representation (10) cast in terms of the history $\check{z}^t$ and **not** a corresponding representation cast in terms of z^t.

5.2 Dynamic Programming and Time Consistency of follower's Problem

Given the sequence $\vec{q}_2$ chosen by the Stackelberg leader in our duopoly model, it turns out that the Stackelberg **follower's** problem is recursive in the *natural* state variables that confront a follower at any time $t \geq 0$.

This means that the follower's plan is time consistent.

To verify these claims, we'll formulate a recursive version of a follower's problem that builds on our recursive representation of the Stackelberg leader's plan and our use of the **Big K, little k** idea.

5.3 Recursive Formulation of a Follower's Problem

We now use what amounts to another "Big K, little k" trick (see rational expectations equilibrium) to formulate a recursive version of a follower's problem cast in terms of an ordinary Bellman equation.

Firm 1, the follower, faces $\{q_{2t}\}_{t=0}^{\infty}$ as a given quantity sequence chosen by the leader and believes that its output price at t satisfies

$$p_t = a_0 - a_1(q_{1t} + q_{2t}), \quad t \geq 0$$

Our challenge is to represent $\{q_{2t}\}_{t=0}^{\infty}$ as a given sequence.

To do so, recall that under the Stackelberg plan, firm 2 sets output according to the q_{2t} component of

$$y_{t+1} = \begin{bmatrix} 1 \\ q_{2t} \\ q_{1t} \\ x_t \end{bmatrix}$$

which is governed by

$$y_{t+1} = (A - BF)y_t$$

To obtain a recursive representation of a $\{q_{2t}\}$ sequence that is exogenous to firm 1, we define a state $\tilde{y}_t$

$$\tilde{y}_t = \begin{bmatrix} 1 \\ q_{2t} \\ \tilde{q}_{1t} \\ \tilde{x}_t \end{bmatrix}$$

that evolves according to

$$\tilde{y}_{t+1} = (A - BF)\tilde{y}_t$$

subject to the initial condition $\tilde{q}_{10} = q_{10}$ and $\tilde{x}_0 = x_0$ where $x_0 = -P_{22}^{-1}P_{21}$ as stated above.

Firm 1's state vector is

$$X_t = \begin{bmatrix} \tilde{y}_t \\ q_{1t} \end{bmatrix}$$

It follows that the follower firm 1 faces law of motion

$$\begin{bmatrix} \tilde{y}_{t+1} \\ q_{1t+1} \end{bmatrix} = \begin{bmatrix} A - BF & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ q_{1t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_t \tag{11}$$

This specfification assures that from the point of the view of a firm 1, q_{2t} is an exogenous process.

Here

- $\tilde{q}_{1t}, \tilde{x}_t$ play the role of **Big K**.
- q_{1t}, x_t play the role of **little k**.

The time t component of firm 1's objective is

$$\tilde{X}_t' \tilde{R} x_t - x_t^2 \tilde{Q} = \begin{bmatrix} 1 \\ q_{2t} \\ \tilde{q}_{1t} \\ \tilde{x}_t \\ q_{1t} \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{a_0}{2} \\ 0 & 0 & 0 & 0 & -\frac{a_1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{a_0}{2} & -\frac{a_1}{2} & 0 & 0 & -a_1 \end{bmatrix} \begin{bmatrix} 1 \\ q_{2t} \\ \tilde{q}_{1t} \\ \tilde{x}_t \\ q_{1t} \end{bmatrix} - \gamma x_t^2$$

Firm 1's optimal decision rule is

$$x_t = -\tilde{F} X_t$$

and it's state evolves according to

$$\tilde{X}_{t+1} = (\tilde{A} - \tilde{B}\tilde{F}) X_t$$

under its optimal decision rule.

Later we shall compute $\tilde{F}$ and verify that when we set

$$X_0 = \begin{bmatrix} 1 \\ q_{20} \\ q_{10} \\ x_0 \\ q_{10} \end{bmatrix}$$

we recover

$$x_0 = -\tilde{F} \tilde{X}_0$$

which will verify that we have properly set up a recursive representation of the follower's problem facing the Stackelberg leader's $\vec{q}_2$.

5.4 Time Consistency of Follower's Plan

Since the follower can solve its problem using dynamic programming its problem is recursive in what for it are the **natural state variables**, namely

$$\begin{bmatrix} 1 \\ q_{2t} \\ \tilde{q}_{10} \\ \tilde{x}_0 \end{bmatrix}$$

It follows that the follower's plan is time consistent.

6 Computing the Stackelberg Plan

Here is our code to compute a Stackelberg plan via a linear-quadratic dynamic program as outlined above

```
In [1]: using InstantiateFromURL
        # optionally add arguments to force installation: instantiate = true,
 ↪precompile = true
        github_project("QuantEcon/quantecon-notebooks-julia", version = "0.8.0")
```

```
In [2]: using QuantEcon, Plots, LinearAlgebra, Statistics, Parameters, Random
        gr(fmt = :png);
```

We define named tuples and default values for the model and solver settings, and instantiate one copy of each

```
In [3]: model = @with_kw (a0 = 10,
                          a1 = 2,
                          β = 0.96,
                          γ = 120.,
                          n = 300)

        # things like tolerances, etc.
        settings = @with_kw (tol0 = 1e-8,
                             tol1 = 1e-16,
                             tol2 = 1e-2)

        defaultModel = model();
        defaultSettings = settings();
```

Now we can compute the actual policy using the LQ routine from QuantEcon.jl

```
In [4]: @unpack a0, a1, β, γ, n = defaultModel
        @unpack tol0, tol1, tol2 = defaultSettings

        βs = [β^x for x = 0:n-1]
        Alhs = I + zeros(4, 4);
        Alhs[4, :] = [β * a0 / (2 * γ), -β * a1 / (2 * γ), -β * a1 / γ, β] # Euler
 ↪equation
```

```
coefficients
Arhs = I + zeros(4, 4);
Arhs[3, 4] = 1.;
Alhsinv = inv(Alhs);

A = Alhsinv * Arhs;
B = Alhsinv * [0, 1, 0, 0,];
R = [0 -a0/2 0 0; -a0/2 a1 a1/2 0; 0 a1/2 0 0; 0 0 0 0];
Q = γ;
lq = QuantEcon.LQ(Q, R, A, B, bet=β);
P, F, d = stationary_values(lq)

P22 = P[4:end, 4:end];
P21 = P[4:end, 1:3];
P22inv = inv(P22);
H_0_0 = -P22inv * P21;

# simulate forward
π_leader = zeros(n);
z0 = [1, 1, 1];
x0 = H_0_0 * z0;
y0 = vcat(z0, x0);

Random.seed!(1) # for reproducibility
yt, ut = compute_sequence(lq, y0, n);
π_matrix = R + F' * Q * F;

for t in 1:n
    π_leader[t] = -(yt[:, t]' * π_matrix * yt[:, t]);
end

println("Computed policy for Stackelberg leader: $F")
```

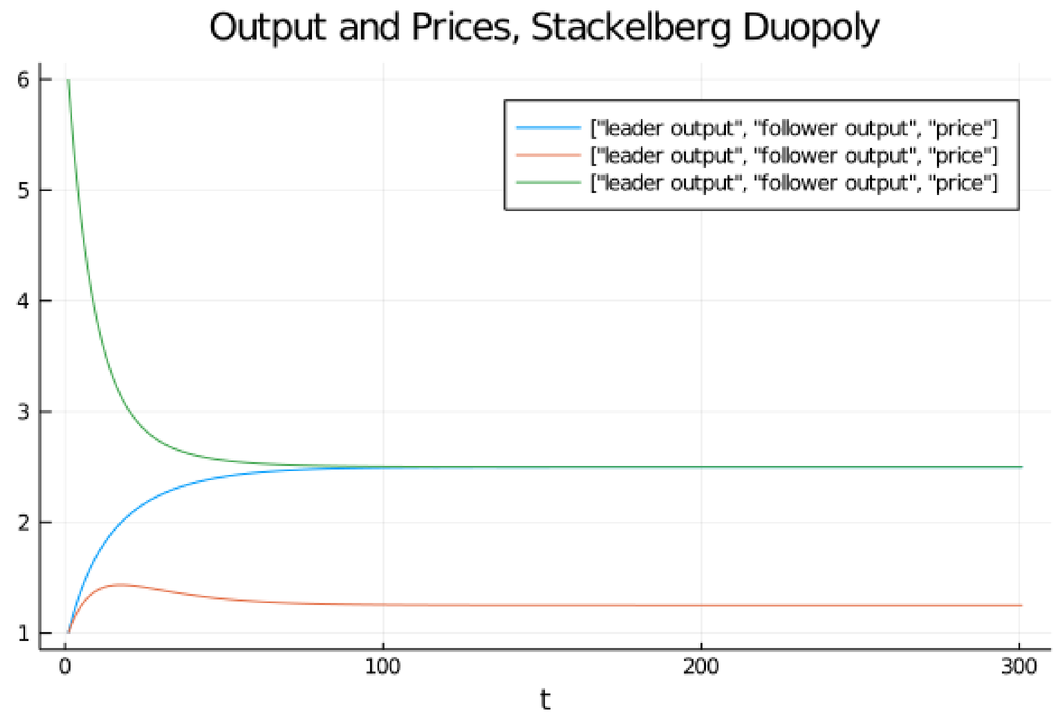
```
Computed policy for Stackelberg leader: [-1.5800445387726552 0.294613127470314
0.6748093760774969 6.539705936147513]
```

6.1 Implied Time Series for Price and Quantities

The following code plots the price and quantities

In [5]:
```
q_leader = yt[2, 1:end];
q_follower = yt[3, 1:end];
q = q_leader + q_follower;
p = a0 .- a1*q;

plot(1:n+1, [q_leader, q_follower, p],
    title = "Output and Prices, Stackelberg Duopoly",
    labels = ["leader output", "follower output", "price"],
    xlabel = "t")
```

Out[5]:



6.2 Value of Stackelberg Leader

We'll compute the present value earned by the Stackelberg leader.

We'll compute it two ways (they give identical answers – just a check on coding and thinking)

In [6]:
```
v_leader_forward = sum(βs .* π_leader);
v_leader_direct = -yt[:, 1]' * P * yt[:, 1];

println("v_leader_forward (forward sim) is $v_leader_forward")
println("v_leader_direct is $v_leader_direct")
```

```
   v_leader_forward (forward sim) is 150.0316212532548
v_leader_direct is 150.03237147548847
```

In [7]:
```
# manually check whether P is an approximate fixed point
P_next = (R + F' * Q * F + β * (A - B * F)' * P * (A - B * F));
all(P - P_next .< tol0)
```

Out[7]: true

In [8]:
```
# manually checks whether two different ways of computing the
# value function give approximately the same answer
v_expanded = -((y0' * R * y0 + ut[:, 1]' * Q * ut[:, 1] +
           β * (y0' * (A - B * F)' * P * (A - B * F) * y0)));
(v_leader_direct - v_expanded < tol0)[1, 1]
```

Out[8]: true

7 Exhibiting Time Inconsistency of Stackelberg Plan

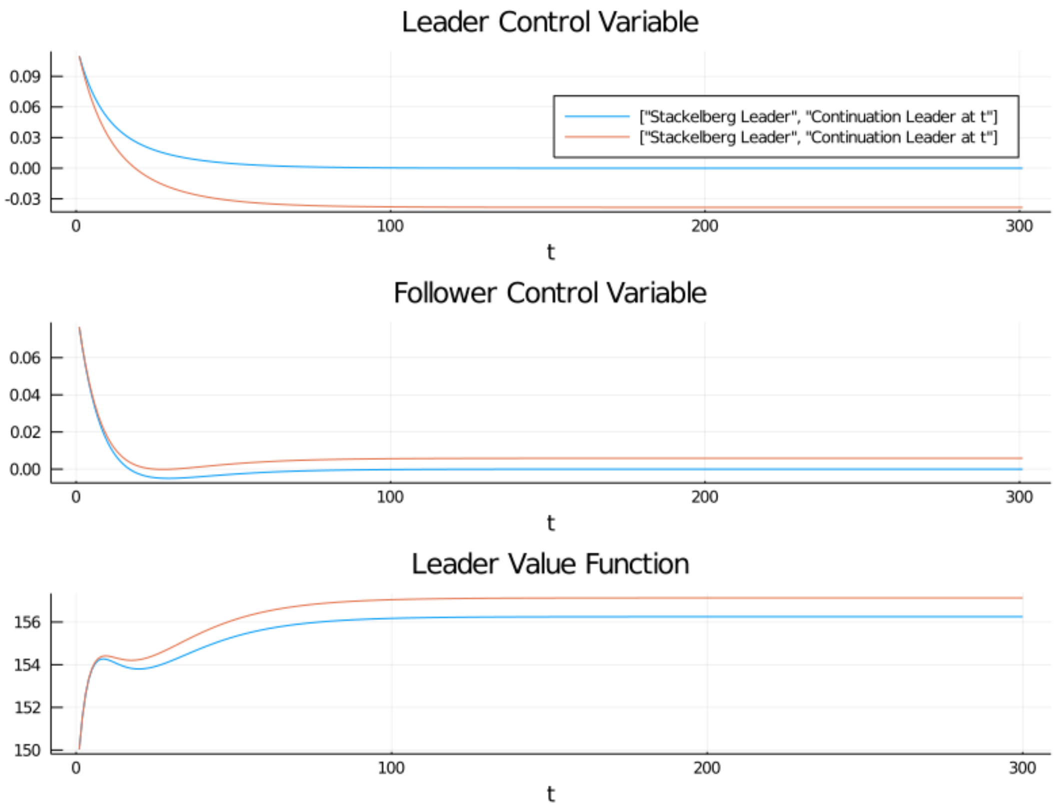
In the code below we compare two values

- the continuation value $-y_t P y_t$ earned by a continuation Stackelberg leader who inherits state y_t at t
- the value of a **reborn Stackelberg leader** who inherits state z_t at t and sets $x_t = -P_{22}^{-1} P_{21}$

The difference between these two values is a tell-tale time of the time inconsistency of the Stackelberg plan

In [9]:
```
# Compute value function over time with reset at time t
vt_leader = zeros(n);
vt_reset_leader = similar(vt_leader);

yt_reset = copy(yt)
yt_reset[end, :] = (H_0_0 * yt[1:3, :])

for t in 1:n
    vt_leader[t] = -yt[:, t]' * P * yt[:, t]
    vt_reset_leader[t] = -yt_reset[:, t]' * P * yt_reset[:, t]
end

p1 = plot(1:n+1, [(-F * yt)', (-F * yt_reset)'], labels = ["Stackelberg␣
↪Leader",
"Continuation Leader at t"],
        title = "Leader Control Variable", xlabel = "t");
p2 = plot(1:n+1, [yt[4, :], yt_reset[4, :]], title = "Follower Control␣
↪Variable", xlabel
= "t", legend = false);
p3 = plot(1:n, [vt_leader, vt_reset_leader], legend = false,
            xlabel = "t", title = "Leader Value Function");
plot(p1, p2, p3, layout = (3, 1), size = (800, 600))
```

Out[9]:



8 Recursive Formulation of the Follower's Problem

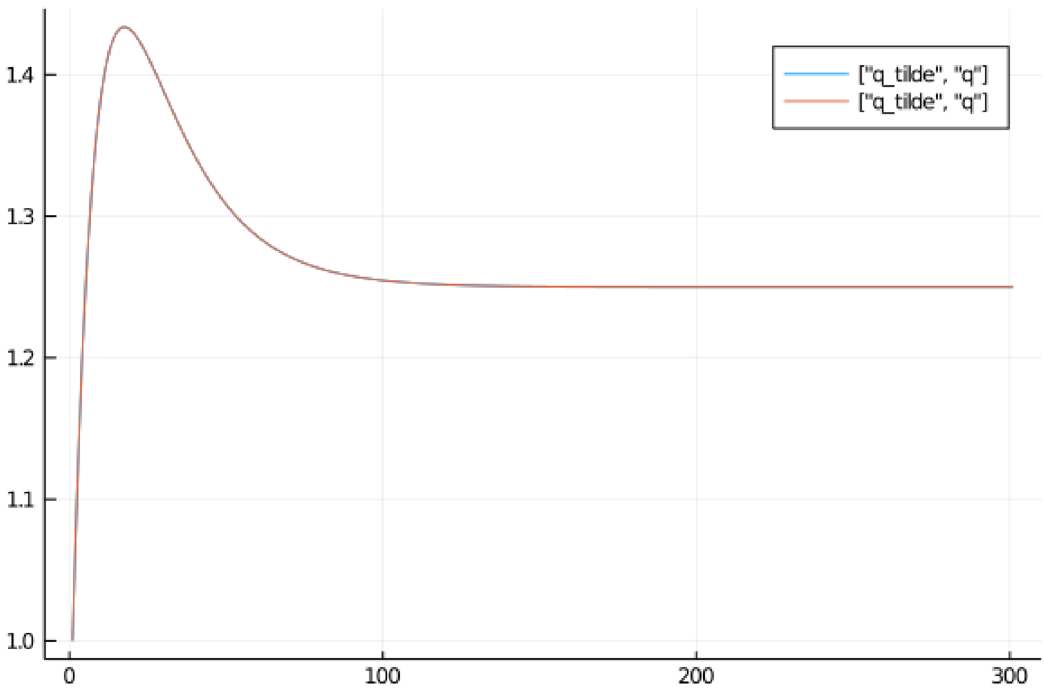
We now formulate and compute the recursive version of the follower's problem.

We check that the recursive **Big** K **, little** k formulation of the follower's problem produces the same output path $\vec{q}_1$ that we computed when we solved the Stackelberg problem

In [10]:
```
Ã = I + zeros(5, 5);
Ã[1:4, 1:4] .= A - B * F;
R̃ = [0 0 0 0 -a0/2; 0 0 0 0 a1/2; 0 0 0 0 0; 0 0 0 0 0; -a0/2 a1/2 0 0 a1];
Q̃ = Q;
B̃ = [0, 0, 0, 0, 1];

lq_tilde = QuantEcon.LQ(Q̃, R̃, Ã, B̃, bet=β);
P̃, F̃, d̃ = stationary_values(lq_tilde);
y0_tilde = vcat(y0, y0[3]);
yt_tilde = compute_sequence(lq_tilde, y0_tilde, n)[1];
```

In [11]:
```
# checks that the recursive formulation of the follower's problem gives
# the same solution as the original Stackelberg problem
plot(1:n+1, [yt_tilde[5, :], yt_tilde[3, :]], labels = ["q_tilde", "q"])
```

Out[11]:



Note: Variables with `_tilde` are obtained from solving the follower's problem – those without are from the Stackelberg problem.

```
In [12]: # maximum absolute difference in quantities over time between the first
  ↪and second
         solution methods
         max(abs(yt_tilde[5] - yt_tilde[3]))
```

```
Out[12]: 0.0
```

```
In [13]: # x0 == x0_tilde
         yt[:, 1][end] - (yt_tilde[:, 2] - yt_tilde[:, 1])[end] < tol0
```

```
Out[13]: true
```

8.1 Explanation of Alignment

If we inspect the coefficients in the decision rule $-\tilde{F}$, we can spot the reason that the follower chooses to set $x_t = \tilde{x}_t$ when it sets $x_t = -\tilde{F}X_t$ in the recursive formulation of the follower problem.

Can you spot what features of $\tilde{F}$ imply this?

Hint: remember the components of X_t

```
In [14]: F̃ # policy function in the follower's problem
```

```
Out[14]: 1×5 Array{Float64,2}:
  2.5489e-17  -3.18612e-18  -0.103187  -1.0  0.103187
```

In [15]:
```julia
P # value function in the Stackelberg problem
```

Out[15]:
```
4×4 Array{Float64,2}:
   963.541  -194.605   -511.622   -5258.23
  -194.605    37.3536    81.9771    784.765
  -511.622    81.9771   247.343    2517.05
 -5258.23    784.765   2517.05    25556.2
```

In [16]:
```julia
P̃ # value function in the follower's problem
```

Out[16]:
```
5×5 Array{Float64,2}:
 -18.1991     2.58003     15.6049     151.23      -5.0
   2.58003   -0.969466    -5.26008    -50.9764     1.0
  15.6049    -5.26008    -32.2759    -312.792    -12.3824
 151.23     -50.9764    -312.792    -3031.33    -120.0
  -5.0        1.0        -12.3824    -120.0       14.3824
```

In [17]:
```julia
# manually check that P is an approximate fixed point
all((P  - ((R + F' * Q * F) + β * (A - B * F)' * P * (A - B * F)) .< tol0))
```

Out[17]:
```
true
```

In [18]:
```julia
# compute `P_guess` using `F_tilde_star`
F̃_star = -[0, 0, 0, 1, 0]';
P_guess = zeros(5, 5);

for i in 1:1000
    P_guess = ((R̃ + F̃_star' * Q̃ * F̃_star) +
               β * (Ã - B̃ * F̃_star)' * P_guess
               * (Ã - B̃ * F̃_star));
end
```

In [19]:
```julia
# value function in the follower's problem
-(y0_tilde' * P̃ * y0_tilde)[1, 1]
```

Out[19]:
```
112.65590740578102
```

In [20]:
```julia
# value function using P_guess
-(y0_tilde' * P_guess * y0_tilde)[1, 1]
```

Out[20]:
```
112.65590740578097
```

In [21]:
```julia
# c policy using policy iteration algorithm
F_iter = (β * inv(Q + β * B̃' * P_guess * B̃)
      * B̃' * P_guess * Ã);
P_iter = zeros(5, 5);
dist_vec = zeros(5, 5);
```

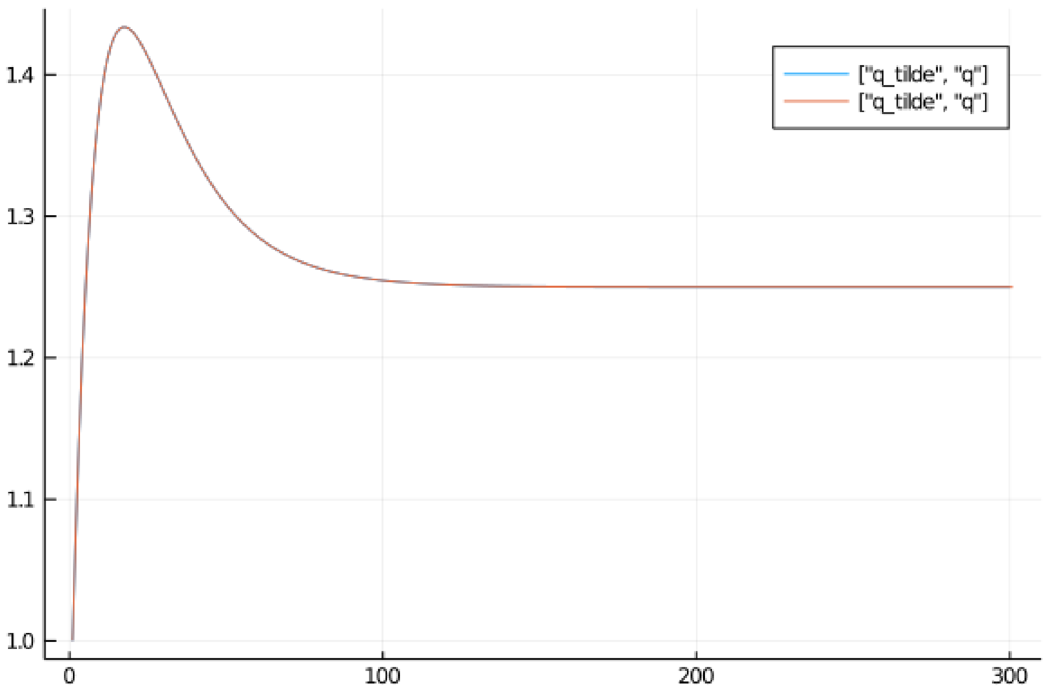
```julia
    for i in 1:100
        # compute P_iter
        dist_vec = similar(P_iter)
        for j in 1:1000
            P_iter = (R̃ + F_iter' * Q * F_iter) + β *
                        (Ã - B̃ * F_iter)' * P_iter *
                        (Ã - B̃ * F_iter);

            # update F_iter
            F_iter = β * inv(Q + β * B̃' * P_iter * B̃) *
                          B̃' * P_iter * Ã;


            dist_vec = P_iter - ((R̃ + F_iter' * Q * F_iter) +
                    β * (Ã - B̃ * F_iter)' * P_iter *
                    (Ã - B̃ * F_iter));
        end
    end

    if maximum(abs.(dist_vec)) < 1e-8
       dist_vec2 = F_iter - (β * inv(Q + β * B̃' * P_iter * B̃) * B̃' * P_iter * Ã)
            if maximum(abs.(dist_vec2)) < 1e-8
                @show F_iter
            else
               println("The policy didn't converge: try increasing the number of outer loop
    iterations")
            end
    else
        println("The policy didn't converge: try increasing the number of inner loop
    iterations")
    end
```

```
F_iter = [0.0 -1.474514954580286e-17 -0.1031865014522383 -1.0000000000000007
0.10318650145223823]
```

Out[21]:
```
1×5 Adjoint{Float64,Array{Float64,1}}:
 0.0  -1.47451e-17  -0.103187  -1.0  0.103187
```

In [22]:
```julia
yt_tilde_star = zeros(n, 5);
yt_tilde_star[1, :] = y0_tilde;

for t in 1:n-1
    yt_tilde_star[t+1, :] = (Ã - B̃ * F̃_star) * yt_tilde_star[t, :];
end

plot([yt_tilde_star[:, 5], yt_tilde[3, :]], labels = ["q_tilde", "q"])
```

Out[22]:




```
In [23]: maximum(abs.(yt_tilde_star[:, 5] - yt_tilde[3, 1:end-1]))
```

```
Out[23]: 0.0
```

9 Markov Perfect Equilibrium

The **state** vector is

$$z_t = \begin{bmatrix} 1 \\ q_{2t} \\ q_{1t} \end{bmatrix}$$

and the state transition dynamics are

$$z_{t+1} = Az_t + B_1 v_{1t} + B_2 v_{2t}$$

where A is a 3×3 identity matrix and

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The Markov perfect decision rules are

$$v_{1t} = -F_1 z_t, \quad v_{2t} = -F_2 z_t$$

and in the Markov perfect equilibrium the state evolves according to

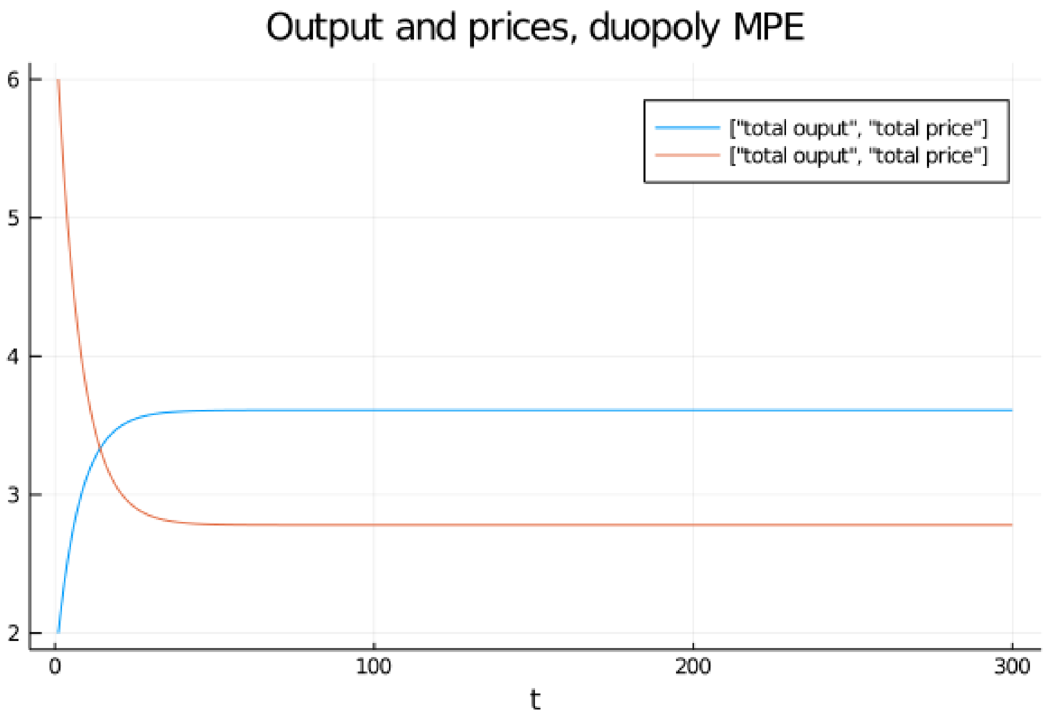
$$z_{t+1} = (A - B_1 F_1 - B_2 F_2) z_t$$

In [24]:
```
# in LQ form
A = I + zeros(3, 3);
B1 = [0, 0, 1];
B2 = [0, 1, 0];
R1 = [0 0 -a0/2; 0 0 a1/2; -a0/2 a1/2 a1];
R2 = [0 -a0/2 0; -a0/2 a1 a1/2; 0 a1/2 0];
Q1 = Q2 = γ;
S1 = S2 = W1 = W2 = M1 = M2 = 0.;

# solve using nnash from QE
F1, F2, P1, P2 = nnash(A, B1, B2, R1, R2, Q1, Q2,
                        S1, S2, W1, W2, M1, M2,
                        beta = β,
                        tol = tol1);

# simulate forward
AF = A - B1 * F1 - B2 * F2;
z = zeros(3, n);
z[:, 1] .= 1;
for t in 1:n-1
    z[:, t+1] = AF * z[:, t]
end

println("Policy for F1 is $F1")
println("Policy for F2 is $F2")
```

```
Policy for F1 is [-0.22701362843207126 0.031298741184441059 0.09447112842804818]
Policy for F2 is [-0.22701362843207126 0.09447112842804818 0.031298741184441059]
```

In [25]:
```
q1 = z[2, :];
q2 = z[3, :];
q = q1 + q2; # total output, MPE
p = a0 .- a1 * q; # total price, MPE
plot([q, p], labels = ["total ouput", "total price"], title = "Output and
↪prices,
duopoly MPE", xlabel = "t")
```

Out[25]:



In [26]:
```
# computes the maximum difference in quantities across firms
maximum(abs.(q1 - q2))
```

Out[26]: 8.881784197001252e-16

In [27]:
```
# compute values
u1 = -F1 * z;
u2 = -F2 * z;
π_1 = (p .* q1)' - γ * u1.^2;
π_2 = (p .* q2)' - γ * u2.^2;

v1_forward = π_1 * βs;
v2_forward = π_2 * βs;

v1_direct = -z[:, 1]' * P1 * z[:, 1];
v2_direct = -z[:, 1]' * P2 * z[:, 1];

println("Firm 1: Direct is $v1_direct, Forward is $(v1_forward[1])");
println("Firm 2: Direct is $v2_direct, Forward is $(v2_forward[1])");
```

```
Firm 1: Direct is 133.3295555721595, Forward is 133.33033197956638
Firm 2: Direct is 133.32955557215945, Forward is 133.33033197956638
```

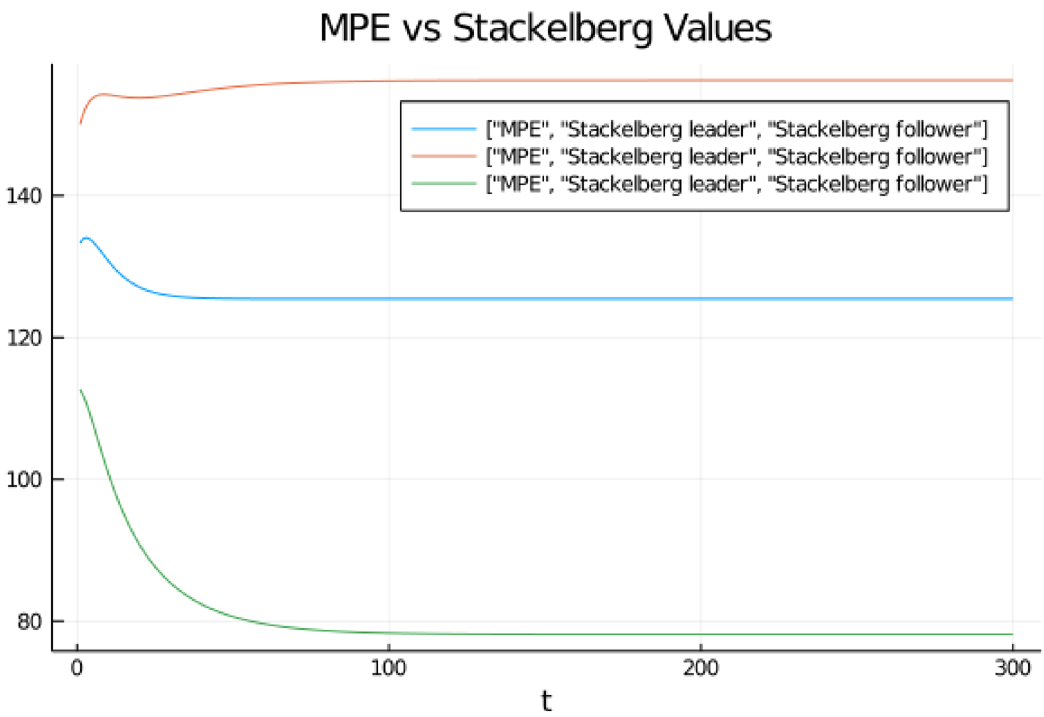
In [28]:
```
# sanity check
Λ_1 = A - B2 * F2;
lq1 = QuantEcon.LQ(Q1, R1, Λ_1, B1, bet = β);
P1_ih, F1_ih, d = stationary_values(lq1);

v2_direct_alt = -z[:, 1]' * P1_ih * z[:, 1] + d;
all(abs.(v2_direct - v2_direct_alt) < tol2)
```

Out[28]: true

10 MPE vs. Stackelberg

In [29]:
```
vt_MPE = zeros(n);
vt_follower = zeros(n);

for t in 1:n
    vt_MPE[t] = -z[:, t]' * P1 * z[:, t];
    vt_follower[t] = -yt_tilde[:, t]' * P̃ * yt_tilde[:, t];
end

plot([vt_MPE, vt_leader, vt_follower], labels = ["MPE", "Stackelberg leader",
        "Stackelberg follower"], title = "MPE vs Stackelberg Values",
        xlabel = "t")
```

Out[29]:



In [30]:
```
# display values
println("vt_leader(y0) = $(vt_leader[1])");
println("vt_follower(y0) = $(vt_follower[1])")
println("vt_MPE(y0) = $(vt_MPE[1])");
```

```
vt_leader(y0) = 150.03237147548847
vt_follower(y0) = 112.65590740578102
vt_MPE(y0) = 133.3295555721595
```

In [31]:
```
# total difference in value b/t Stackelberg and MPE
vt_leader[1] + vt_follower[1] - 2*vt_MPE[1]
```

Out[31]: -3.9708322630494877