Optimal Taxation without State-Contingent Debt

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2 Overview

In an earlier lecture we described a model of optimal taxation with state-contingent debt due to Robert E. Lucas, Jr., and Nancy Stokey [3].

Aiyagari, Marcet, Sargent, and Seppälä [1] (hereafter, AMSS) studied optimal taxation in a model without state-contingent debt.

In this lecture, we

- describe assumptions and equilibrium concepts
- solve the model
- implement the model numerically
- conduct some policy experiments
- compare outcomes with those in a corresponding complete-markets model

We begin with an introduction to the model.

2.1 Setup

```
In [1]: using InstantiateFromURL
        # optionally add arguments to force installation: instantiate = true,
 ↪precompile = true
        github_project("QuantEcon/quantecon-notebooks-julia", version = "0.8.0")
```

```
In [2]: using LinearAlgebra, Statistics
```

3 Competitive Equilibrium with Distorting Taxes

Many but not all features of the economy are identical to those of the Lucas-Stokey economy.

Let's start with things that are identical.

For $t \geq 0$, a history of the state is represented by $s^t = [s_t, s_{t-1}, \dots, s_0]$.

Government purchases $g(s)$ are an exact time-invariant function of s.

Let $c_t(s^t)$, $\ell_t(s^t)$, and $n_t(s^t)$ denote consumption, leisure, and labor supply, respectively, at history s^t at time t.

Each period a representative household is endowed with one unit of time that can be divided between leisure ℓ_t and labor n_t:

$$n_t(s^t) + \ell_t(s^t) = 1 \tag{1}$$

Output equals $n_t(s^t)$ and can be divided between consumption $c_t(s^t)$ and $g(s_t)$

$$c_t(s^t) + g(s_t) = n_t(s^t) \tag{2}$$

Output is not storable.

The technology pins down a pre-tax wage rate to unity for all t, s^t.

A representative household's preferences over $\{c_t(s^t), \ell_t(s^t)\}_{t=0}^{\infty}$ are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u[c_t(s^t), \ell_t(s^t)] \tag{3}$$

where

- $\pi_t(s^t)$ is a joint probability distribution over the sequence s^t, and
- the utility function u is increasing, strictly concave, and three times continuously differentiable in both arguments

The government imposes a flat rate tax $\tau_t(s^t)$ on labor income at time t, history s^t.

Lucas and Stokey assumed that there are complete markets in one-period Arrow securities; also see smoothing models.

It is at this point that AMSS [1] modify the Lucas and Stokey economy.

AMSS allow the government to issue only one-period risk-free debt each period.

Ruling out complete markets in this way is a step in the direction of making total tax collections behave more like that prescribed in [2] than they do in [3].

3.1 Risk-free One-Period Debt Only

In period t and history s^t, let

- $b_{t+1}(s^t)$ be the amount of the time $t+1$ consumption good that at time t the government promised to pay.
- $R_t(s^t)$ be the gross interest rate on risk-free one-period debt between periods t and $t+1$.

- $T_t(s^t)$ be a nonnegative lump-sum transfer to the representative household Section **??**.

That $b_{t+1}(s^t)$ is the same for all realizations of s_{t+1} captures its *risk-free* character.

The market value at time t of government debt maturing at time $t+1$ equals $b_{t+1}(s^t)$ divided by $R_t(s^t)$.

The government's budget constraint in period t at history s^t is

$$\begin{aligned} b_t(s^{t-1}) &= \tau_t^n(s^t)n_t(s^t) - g_t(s_t) - T_t(s^t) + \frac{b_{t+1}(s^t)}{R_t(s^t)} \\ &\equiv z(s^t) + \frac{b_{t+1}(s^t)}{R_t(s^t)}, \end{aligned} \tag{4}$$

where $z(s^t)$ is the net-of-interest government surplus.

To rule out Ponzi schemes, we assume that the government is subject to a **natural debt limit** (to be discussed in a forthcoming lecture).

The consumption Euler equation for a representative household able to trade only one-period risk-free debt with one-period gross interest rate $R_t(s^t)$ is

$$\frac{1}{R_t(s^t)} = \sum_{s^{t+1}|s^t} \beta\pi_{t+1}(s^{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}$$

Substituting this expression into the government's budget constraint (4) yields:

$$b_t(s^{t-1}) = z(s^t) + \beta \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}\; b_{t+1}(s^t) \tag{5}$$

Components of $z(s^t)$ on the right side depend on s^t, but the left side is required to depend on s^{t-1} only.

This is what it means for one-period government debt to be risk-free.

Therefore, the sum on the right side of equation (5) also has to depend only on s^{t-1}.

This requirement will give rise to **measurability constraints** on the Ramsey allocation to be discussed soon.

If we replace $b_{t+1}(s^t)$ on the right side of equation (5) by the right side of next period's budget constraint (associated with a particular realization s_t) we get

$$b_t(s^{t-1}) = z(s^t) + \sum_{s^{t+1}|s^t} \beta\pi_{t+1}(s^{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)}\;\left[z(s^{t+1}) + \frac{b_{t+2}(s^{t+1})}{R_{t+1}(s^{t+1})}\right]$$

After making similar repeated substitutions for all future occurrences of government indebtedness, and by invoking the natural debt limit, we arrive at:

$$b_t(s^{t-1}) = \sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j\pi_{t+j}(s^{t+j}|s^t)\frac{u_c(s^{t+j})}{u_c(s^t)}\; z(s^{t+j}) \tag{6}$$

Now let's

- substitute the resource constraint into the net-of-interest government surplus, and
- use the household's first-order condition $1-\tau_t^n(s^t) = u_\ell(s^t)/u_c(s^t)$ to eliminate the labor tax rate

so that we can express the net-of-interest government surplus $z(s^t)$ as

$$z(s^t) = \left[1 - \frac{u_\ell(s^t)}{u_c(s^t)}\right] [c_t(s^t) + g_t(s_t)] - g_t(s_t) - T_t(s^t)\,. \tag{7}$$

If we substitute the appropriate versions of right side of (7) for $z(s^{t+j})$ into equation (6), we obtain a sequence of *implementability constraints* on a Ramsey allocation in an AMSS economy.

Expression (6) at time $t = 0$ and initial state s^0 was also an *implementability constraint* on a Ramsey allocation in a Lucas-Stokey economy:

$$b_0(s^{-1}) = \mathbb{E}_0 \sum_{j=0}^{\infty} \beta^j \frac{u_c(s^j)}{u_c(s^0)}\, z(s^j) \tag{8}$$

Indeed, it was the *only* implementability constraint there.

But now we also have a large number of additional implementability constraints

$$b_t(s^{t-1}) = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{u_c(s^{t+j})}{u_c(s^t)}\, z(s^{t+j}) \tag{9}$$

Equation (9) must hold for each s^t for each $t \geq 1$.

3.2 Comparison with Lucas-Stokey Economy

The expression on the right side of (9) in the Lucas-Stokey (1983) economy would equal the present value of a continuation stream of government surpluses evaluated at what would be competitive equilibrium Arrow-Debreu prices at date t.

In the Lucas-Stokey economy, that present value is measurable with respect to s^t.

In the AMSS economy, the restriction that government debt be risk-free imposes that that same present value must be measurable with respect to s^{t-1}.

In a language used in the literature on incomplete markets models, it can be said that the AMSS model requires that at each (t, s^t) what would be the present value of continuation government surpluses in the Lucas-Stokey model must belong to the **marketable subspace** of the AMSS model.

3.3 Ramsey Problem Without State-contingent Debt

After we have substituted the resource constraint into the utility function, we can express the Ramsey problem as being to choose an allocation that solves

$$\max_{\{c_t(s^t), b_{t+1}(s^t)\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t(s^t), 1 - c_t(s^t) - g_t(s_t)\right)$$

where the maximization is subject to

$$\mathbb{E}_0 \sum_{j=0}^{\infty} \beta^j \frac{u_c(s^j)}{u_c(s^0)} \; z(s^j) \geq b_0(s^{-1}) \tag{10}$$

and

$$\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{u_c(s^{t+j})}{u_c(s^t)} \; z(s^{t+j}) = b_t(s^{t-1}) \quad \forall \, s^t \tag{11}$$

given $b_0(s^{-1})$.

3.3.1 Lagrangian Formulation

Let $\gamma_0(s^0)$ be a nonnegative Lagrange multiplier on constraint (10).

As in the Lucas-Stokey economy, this multiplier is strictly positive when the government must resort to distortionary taxation; otherwise it equals zero.

A consequence of the assumption that there are no markets in state-contingent securities and that a market exists only in a risk-free security is that we have to attach stochastic processes $\{\gamma_t(s^t)\}_{t=1}^{\infty}$ of Lagrange multipliers to the implementability constraints (11).

Depending on how the constraints bind, these multipliers can be positive or negative:

$$\begin{aligned} \gamma_t(s^t) \;&\geq\; (\leq)\; 0 \quad \text{if the constraint binds in this direction} \\ \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{u_c(s^{t+j})}{u_c(s^t)} \; z(s^{t+j}) \;&\geq\; (\leq)\; b_t(s^{t-1}). \end{aligned}$$

A negative multiplier $\gamma_t(s^t) < 0$ means that if we could relax constraint (11), we would like to *increase* the beginning-of-period indebtedness for that particular realization of history s^t.

That would let us reduce the beginning-of-period indebtedness for some other history Section **??**.

These features flow from the fact that the government cannot use state-contingent debt and therefore cannot allocate its indebtedness efficiently across future states.

3.4 Some Calculations

It is helpful to apply two transformations to the Lagrangian.

Multiply constraint (10) by $u_c(s^0)$ and the constraints (11) by $\beta^t u_c(s^t)$.

Then a Lagrangian for the Ramsey problem can be represented as

$$
\begin{aligned}
J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & u\left(c_t(s^t), 1 - c_t(s^t) - g_t(s_t)\right) \\
& + \gamma_t(s^t) \Big[\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u_c(s^{t+j})\, z(s^{t+j}) - u_c(s^t)\, b_t(s^{t-1}) \Big\} \\
= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & u\left(c_t(s^t), 1 - c_t(s^t) - g_t(s_t)\right) \\
& + \Psi_t(s^t)\, u_c(s^t)\, z(s^t) - \gamma_t(s^t)\, u_c(s^t)\, b_t(s^{t-1}) \Big\}
\end{aligned}
\tag{12}
$$

where

$$
\Psi_t(s^t) = \Psi_{t-1}(s^{t-1}) + \gamma_t(s^t) \quad \text{and} \quad \Psi_{-1}(s^{-1}) = 0 \tag{13}
$$

In (12), the second equality uses the law of iterated expectations and Abel's summation formula (also called *summation by parts*, see this page).

First-order conditions with respect to $c_t(s^t)$ can be expressed as

$$
\begin{aligned}
u_c(s^t) - u_\ell(s^t) + \Psi_t(s^t) \left\{ [u_{cc}(s^t) - u_{c\ell}(s^t)]\, z(s^t) + u_c(s^t)\, z_c(s^t) \right\} & \\
- \gamma_t(s^t)\, [u_{cc}(s^t) - u_{c\ell}(s^t)]\, b_t(s^{t-1}) = 0 &
\end{aligned}
\tag{14}
$$

and with respect to $b_t(s^t)$ as

$$
\mathbb{E}_t \left[\gamma_{t+1}(s^{t+1})\, u_c(s^{t+1}) \right] = 0 \tag{15}
$$

If we substitute $z(s^t)$ from (7) and its derivative $z_c(s^t)$ into first-order condition (14), we find two differences from the corresponding condition for the optimal allocation in a Lucas-Stokey economy with state-contingent government debt.

1. The term involving $b_t(s^{t-1})$ in first-order condition (14) does not appear in the corresponding expression for the Lucas-Stokey economy.

   - This term reflects the constraint that beginning-of-period government indebtedness must be the same across all realizations of next period's state, a constraint that would not be present if government debt could be state contingent.

2. The Lagrange multiplier $\Psi_t(s^t)$ in first-order condition (14) may change over time in response to realizations of the state, while the multiplier Φ in the Lucas-Stokey economy is time invariant.

We need some code from our an earlier lecture on optimal taxation with state-contingent debt sequential allocation implementation:

```
In [3]: using QuantEcon, NLsolve, NLopt

        import QuantEcon.simulate

        mutable struct Model{TF <: AbstractFloat,
                            TM <: AbstractMatrix{TF},
                            TV <: AbstractVector{TF}}
            β::TF
            Π::TM
            G::TV
            Θ::TV
            transfers::Bool
            U::Function
            Uc::Function
            Ucc::Function
            Un::Function
            Unn::Function
            n_less_than_one::Bool
        end


        struct SequentialAllocation{TP <: Model,
                                    TI <: Integer,
                                    TV <: AbstractVector}
            model::TP
            mc::MarkovChain
            S::TI
            cFB::TV
            nFB::TV
            ΞFB::TV
            zFB::TV
        end


        function SequentialAllocation(model::Model)
            β, Π, G, Θ = model.β, model.Π, model.G, model.Θ
            mc = MarkovChain(Π)
            S = size(Π, 1)   # Number of states
            # Now find the first best allocation
            cFB, nFB, ΞFB, zFB = find_first_best(model, S, 1)

            return SequentialAllocation(model, mc, S, cFB, nFB, ΞFB, zFB)
        end


        function find_first_best(model::Model, S::Integer, version::Integer)
            if version != 1 && version != 2
                throw(ArgumentError("version must be 1 or 2"))
            end
            β, Θ, Uc, Un, G, Π =
                model.β, model.Θ, model.Uc, model.Un, model.G, model.Π
            function res!(out, z)
                c = z[1:S]
                n = z[S+1:end]
                out[1:S] = Θ .* Uc.(c, n) + Un.(c, n)
                out[S+1:end] = Θ .* n .- c .- G
            end
            res = nlsolve(res!, 0.5 * ones(2 * S))
        if converged(res) == false
            error("Could not find first best")
        end

        if version == 1
            cFB = res.zero[1:S]
            nFB = res.zero[S+1:end]
            ΞFB = Uc(cFB, nFB)         # Multiplier on the resource constraint
            zFB = vcat(cFB, nFB, ΞFB)
            return cFB, nFB, ΞFB, zFB
        elseif version == 2
            cFB = res.zero[1:S]
            nFB = res.zero[S+1:end]
            IFB = Uc(cFB, nFB) .* cFB + Un(cFB, nFB) .* nFB
            xFB = \(LinearAlgebra.I - β * Π, IFB)
            zFB = [vcat(cFB[s], xFB[s], xFB) for s in 1:S]
            return cFB, nFB, IFB, xFB, zFB
        end
    end


    function time1_allocation(pas::SequentialAllocation, μ::Real)
        model, S = pas.model, pas.S
        Θ, β, Π, G, Uc, Ucc, Un, Unn =
            model.Θ, model.β, model.Π, model.G,
            model.Uc, model.Ucc, model.Un, model.Unn
        function FOC!(out, z::Vector)
            c = z[1:S]
            n = z[S+1:2S]
            Ξ = z[2S+1:end]
           out[1:S] = Uc.(c, n) - μ * (Ucc.(c, n) .* c + Uc.(c, n)) - Ξ # FOC c
           out[S+1:2S] = Un.(c, n) - μ * (Unn(c, n) .* n .+ Un.(c, n)) + Θ .*
↪Ξ # FOC n
            out[2S+1:end] = Θ .* n - c .- G # resource constraint
            return out
        end
        # Find the root of the FOC
        res = nlsolve(FOC!, pas.zFB)
        if res.f_converged == false
            error("Could not find LS allocation.")
        end
        z = res.zero
        c, n, Ξ = z[1:S], z[S+1:2S], z[2S+1:end]
        # Now compute x
        I  = Uc(c, n) .* c +  Un(c, n) .* n
        x = \(LinearAlgebra.I - β * model.Π, I)
        return c, n, x, Ξ
    end


    function time0_allocation(pas::SequentialAllocation,
                              B_::AbstractFloat, s_0::Integer)
        model = pas.model
        Π, Θ, G, β = model.Π, model.Θ, model.G, model.β
        Uc, Ucc, Un, Unn =
            model.Uc, model.Ucc, model.Un, model.Unn
```

```julia
    # First order conditions of planner's problem
    function FOC!(out, z)
        μ, c, n, Ξ = z[1], z[2], z[3], z[4]
        xprime = time1_allocation(pas, μ)[3]
        out .= vcat(
        Uc(c, n) .* (c - B_) + Un(c, n) .* n + β * dot(Π[s_0, :], xprime),
        Uc(c, n) .- μ * (Ucc(c, n) .* (c - B_) + Uc(c, n)) .- Ξ,
        Un(c, n) .- μ * (Unn(c, n) .* n + Un(c, n)) + Θ[s_0] .* Ξ,
        (Θ .* n .- c .- G)[s_0]
        )
    end

    # Find root
    res = nlsolve(FOC!, [0.0, pas.cFB[s_0], pas.nFB[s_0], pas.ΞFB[s_0]])
    if res.f_converged == false
        error("Could not find time 0 LS allocation.")
    end
    return (res.zero...,)
end


function time1_value(pas::SequentialAllocation, μ::Real)
    model = pas.model
    c, n, x, Ξ = time1_allocation(pas, μ)
    U_val = model.U.(c, n)
    V = \(LinearAlgebra.I - model.β*model.Π, U_val)
    return c, n, x, V
end


function T(model::Model, c::Union{Real,Vector}, n::Union{Real,Vector})
    Uc, Un = model.Uc.(c, n), model.Un.(c, n)
    return 1. .+ Un./(model.Θ .* Uc)
end


function simulate(pas::SequentialAllocation,
                  B_::AbstractFloat, s_0::Integer,
                  T::Integer,
                  sHist::Union{Vector, Nothing}=nothing)

    model = pas.model
    Π, β, Uc = model.Π, model.β, model.Uc

    if isnothing(sHist)
        sHist = QuantEcon.simulate(pas.mc, T, init=s_0)
    end
    cHist = zeros(T)
    nHist = zeros(T)
    Bhist = zeros(T)
    THist = zeros(T)
    μHist = zeros(T)
    RHist = zeros(T-1)
    # time 0
    μ, cHist[1], nHist[1], _  = time0_allocation(pas, B_, s_0)
    THist[1] = T(pas.model, cHist[1], nHist[1])[s_0]
    Bhist[1] = B_
    μHist[1] = μ
    # time 1 onward
    for t in 2:T
        c, n, x, Ξ = time1_allocation(pas,μ)
        u_c = Uc(c,n)
        s = sHist[t]
        ΤHist[t] = Τ(pas.model, c, n)[s]
        Eu_c = dot(Π[sHist[t-1],:], u_c)
        cHist[t], nHist[t], Bhist[t] = c[s], n[s], x[s] / u_c[s]
        RHist[t-1] = Uc(cHist[t-1], nHist[t-1]) / (β * Eu_c)
        μHist[t] = μ
    end
    return cHist, nHist, Bhist, ΤHist, sHist, μHist, RHist
end


mutable struct BellmanEquation{TP <: Model,
                                TI <: Integer,
                                TV <: AbstractVector,
                                TM <: AbstractMatrix{TV},
                                TVV <: AbstractVector{TV}}
    model::TP
    S::TI
    xbar::TV
    time_0::Bool
    z0::TM
    cFB::TV
    nFB::TV
    xFB::TV
    zFB::TVV
end


function BellmanEquation(model::Model, xgrid::AbstractVector, policies0::
↪Vector)
    S = size(model.Π, 1) # number of states
    xbar = [minimum(xgrid), maximum(xgrid)]
    time_0 = false
    cf, nf, xprimef = policies0
   z0 = [vcat(cf[s](x), nf[s](x), [xprimef[s, sprime](x) for sprime in 1:S])
                        for x in xgrid, s in 1:S]
    cFB, nFB, IFB, xFB, zFB = find_first_best(model, S, 2)
    return BellmanEquation(model, S, xbar, time_0, z0, cFB, nFB, xFB, zFB)
end


function get_policies_time1(T::BellmanEquation,
                        i_x::Integer, x::AbstractFloat,
                        s::Integer, Vf::AbstractArray)
    model, S = T.model, T.S
    β, Θ, G, Π = model.β, model.Θ, model.G, model.Π
    U, Uc, Un = model.U, model.Uc, model.Un

    function objf(z::Vector, grad)
        c, xprime = z[1], z[2:end]
        n=c+G[s]
        Vprime = [Vf[sprime](xprime[sprime]) for sprime in 1:S]
        return -(U(c, n) + β * dot(Π[s, :], Vprime))
    end
    function cons(z::Vector, grad)
        c, xprime = z[1], z[2:end]
        n=c+G[s]
        return x - Uc(c, n) * c - Un(c, n) * n - β * dot(Π[s, :], xprime)
    end
    lb = vcat(0, T.xbar[1] * ones(S))
    ub = vcat(1 - G[s], T.xbar[2] * ones(S))
    opt = Opt(:LN_COBYLA, length(T.z0[i_x, s])-1)
    min_objective!(opt, objf)
    equality_constraint!(opt, cons)
    lower_bounds!(opt, lb)
    upper_bounds!(opt, ub)
    maxeval!(opt, 300)
    maxtime!(opt, 10)
    init = vcat(T.z0[i_x, s][1], T.z0[i_x, s][3:end])
    for (i, val) in enumerate(init)
        if val > ub[i]
            init[i] = ub[i]
        elseif val < lb[i]
            init[i] = lb[i]
        end
    end
    (minf, minx, ret) = optimize(opt, init)
    T.z0[i_x, s] = vcat(minx[1], minx[1] + G[s], minx[2:end])
    return vcat(-minf, T.z0[i_x, s])
end

function get_policies_time0(T::BellmanEquation,
                            B_::AbstractFloat, s0::Integer, Vf::Array)
    model, S = T.model, T.S
    β, Θ, G, Π = model.β, model.Θ, model.G, model.Π
    U, Uc, Un = model.U, model.Uc, model.Un
    function objf(z, grad)
        c, xprime = z[1], z[2:end]
        n = c+G[s0]
        Vprime = [Vf[sprime](xprime[sprime]) for sprime in 1:S]
        return -(U(c, n) + β * dot(Π[s0, :], Vprime))
    end
    function cons(z::Vector, grad)
        c, xprime = z[1], z[2:end]
        n = c + G[s0]
      return -Uc(c, n) * (c - B_) - Un(c, n) * n - β * dot(Π[s0, :], xprime)
    end
    lb = vcat(0, T.xbar[1] * ones(S))
    ub = vcat(1-G[s0], T.xbar[2] * ones(S))
    opt = Opt(:LN_COBYLA, length(T.zFB[s0])-1)
    min_objective!(opt, objf)
    equality_constraint!(opt, cons)
    lower_bounds!(opt, lb)
    upper_bounds!(opt, ub)
    maxeval!(opt, 300)
    maxtime!(opt, 10)
    init = vcat(T.zFB[s0][1], T.zFB[s0][3:end])
    for (i, val) in enumerate(init)
        if val > ub[i]
            init[i] = ub[i]
        elseif val < lb[i]
            init[i] = lb[i]
            end
        end
        (minf, minx, ret) = optimize(opt, init)
        return vcat(-minf, vcat(minx[1], minx[1]+G[s0], minx[2:end]))
    end
```

```
Out[3]: get_policies_time0 (generic function with 1 method)
```

To analyze the AMSS model, we find it useful to adopt a recursive formulation using techniques like those in our lectures on dynamic Stackelberg models and optimal taxation with state-contingent debt.

4 Recursive Version of AMSS Model

We now describe a recursive formulation of the AMSS economy.

We have noted that from the point of view of the Ramsey planner, the restriction to one-period risk-free securities

- leaves intact the single implementability constraint on allocations (8) from the Lucas-Stokey economy, but
- adds measurability constraints (6) on functions of tails of allocations at each time and history

We now explore how these constraints alter Bellman equations for a time 0 Ramsey planner and for time $t \geq 1$, history s^t continuation Ramsey planners.

4.1 Recasting State Variables

In the AMSS setting, the government faces a sequence of budget constraints

$$\tau_t(s^t)n_t(s^t) + T_t(s^t) + b_{t+1}(s^t)/R_t(s^t) = g_t + b_t(s^{t-1})$$

where $R_t(s^t)$ is the gross risk-free rate of interest between t and $t+1$ at history s^t and $T_t(s^t)$ are nonnegative transfers.

Throughout this lecture, we shall set transfers to zero (for some issues about the limiting behavior of debt, this makes a possibly important difference from AMSS [1], who restricted transfers to be nonnegative).

In this case, the household faces a sequence of budget constraints

$$b_t(s^{t-1}) + (1-\tau_t(s^t))n_t(s^t) = c_t(s^t) + b_{t+1}(s^t)/R_t(s^t) \tag{16}$$

The household's first-order conditions are $u_{c,t} = \beta R_t \mathbb{E}_t u_{c,t+1}$ and $(1-\tau_t)u_{c,t} = u_{l,t}$.

Using these to eliminate R_t and τ_t from budget constraint (16) gives

$$b_t(s^{t-1}) + \frac{u_{l,t}(s^t)}{u_{c,t}(s^t)}n_t(s^t) = c_t(s^t) + \frac{\beta(\mathbb{E}_t u_{c,t+1})b_{t+1}(s^t)}{u_{c,t}(s^t)} \tag{17}$$

or

$$u_{c,t}(s^t)b_t(s^{t-1}) + u_{l,t}(s^t)n_t(s^t) = u_{c,t}(s^t)c_t(s^t) + \beta(\mathbb{E}_t u_{c,t+1})b_{t+1}(s^t) \tag{18}$$

Now define

$$x_t \equiv \beta b_{t+1}(s^t)\mathbb{E}_t u_{c,t+1} = u_{c,t}(s^t)\frac{b_{t+1}(s^t)}{R_t(s^t)} \tag{19}$$

and represent the household's budget constraint at time t, history s^t as

$$\frac{u_{c,t}x_{t-1}}{\beta\mathbb{E}_{t-1}u_{c,t}} = u_{c,t}c_t - u_{l,t}n_t + x_t \tag{20}$$

for $t \geq 1$.

4.2 Measurability Constraints

Write equation (18) as

$$b_t(s^{t-1}) = c_t(s^t) - \frac{u_{l,t}(s^t)}{u_{c,t}(s^t)}n_t(s^t) + \frac{\beta(\mathbb{E}_t u_{c,t+1})b_{t+1}(s^t)}{u_{c,t}} \tag{21}$$

The right side of equation (21) expresses the time t value of government debt in terms of a linear combination of terms whose individual components are measurable with respect to s^t.

The sum of terms on the right side of equation (21) must equal $b_t(s^{t-1})$.

That implies that it is has to be *measurable* with respect to s^{t-1}.

Equations (21) are the *measurablility constraints* that the AMSS model adds to the single time 0 implementation constraint imposed in the Lucas and Stokey model.

4.3 Two Bellman Equations

Let $\Pi(s|s\_)$ be a Markov transition matrix whose entries tell probabilities of moving from state $s\_$ to state s in one period.

Let

- $V(x\_, s\_)$ be the continuation value of a continuation Ramsey plan at $x_{t-1} = x\_, s_{t-1} = s\_$ for $t \geq 1$.
- $W(b, s)$ be the value of the Ramsey plan at time 0 at $b_0 = b$ and $s_0 = s$.

We distinguish between two types of planners:

For $t \geq 1$, the value function for a **continuation Ramsey planner** satisfies the Bellman equation

$$V(x\_, s\_) = \max_{\{n(s),x(s)\}} \sum_s \Pi(s|s\_)\left[u(n(s) - g(s), 1 - n(s)) + \beta V(x(s), s)\right] \tag{22}$$

subject to the following collection of implementability constraints, one for each $s \in S$:

$$\frac{u_c(s)x_-}{\beta\sum_{\tilde{s}}\Pi(\tilde{s}|s_-)u_c(\tilde{s})} = u_c(s)(n(s)-g(s)) - u_l(s)n(s) + x(s) \tag{23}$$

A continuation Ramsey planner at $t \geq 1$ takes $(x_{t-1}, s_{t-1}) = (x_-, s_-)$ as given and before s is realized chooses $(n_t(s_t), x_t(s_t)) = (n(s), x(s))$ for $s \in S$.

The **Ramsey planner** takes (b_0, s_0) as given and chooses (n_0, x_0).

The value function $W(b_0, s_0)$ for the time $t = 0$ Ramsey planner satisfies the Bellman equation

$$W(b_0, s_0) = \max_{n_0, x_0} u(n_0 - g_0, 1 - n_0) + \beta V(x_0, s_0) \tag{24}$$

where maximization is subject to

$$u_{c,0}b_0 = u_{c,0}(n_0 - g_0) - u_{l,0}n_0 + x_0 \tag{25}$$

4.4 Martingale Supercedes State-Variable Degeneracy

Let $\mu(s|s_-)\Pi(s|s_-)$ be a Lagrange multiplier on constraint (23) for state s.

After forming an appropriate Lagrangian, we find that the continuation Ramsey planner's first-order condition with respect to $x(s)$ is

$$\beta V_x(x(s), s) = \mu(s|s_-) \tag{26}$$

Applying the envelope theorem to Bellman equation (22) gives

$$V_x(x_-, s_-) = \sum_s \Pi(s|s_-)\mu(s|s_-)\frac{u_c(s)}{\beta\sum_{\tilde{s}}\Pi(\tilde{s}|s_-)u_c(\tilde{s})} \tag{27}$$

Equations (26) and (27) imply that

$$V_x(x_-, s_-) = \sum_s \left(\Pi(s|s_-)\frac{u_c(s)}{\sum_{\tilde{s}}\Pi(\tilde{s}|s_-)u_c(\tilde{s})}\right) V_x(x(s), s) \tag{28}$$

Equation (28) states that $V_x(x, s)$ is a *risk-adjusted martingale.*

Saying that $V_x(x, s)$ is a risk-adjusted martingale means that $V_x(x, s)$ is a martingale with respect to the probability distribution over s^t sequences that is generated by the *twisted* transition probability matrix:

$$\check{\Pi}(s|s_-) \equiv \Pi(s|s_-)\frac{u_c(s)}{\sum_{\tilde{s}}\Pi(\tilde{s}|s_-)u_c(\tilde{s})}$$

Exercise: Please verify that $\check{\Pi}(s|s_-)$ is a valid Markov transition density, i.e., that its elements are all nonnegative and that for each s_-, the sum over s equals unity.

4.5 Absence of State Variable Degeneracy

Along a Ramsey plan, the state variable $x_t = x_t(s^t, b_0)$ becomes a function of the history s^t and initial government debt b_0.

In Lucas-Stokey model, we found that

- a counterpart to $V_x(x, s)$ is time invariant and equal to the Lagrange multiplier on the Lucas-Stokey implementability constraint
- time invariance of $V_x(x, s)$ is the source of a key feature of the Lucas-Stokey model, namely, state variable degeneracy (i.e., x_t is an exact function of s_t)

That $V_x(x, s)$ varies over time according to a twisted martingale means that there is no state-variable degeneracy in the AMSS model.

In the AMSS model, both x and s are needed to describe the state.

This property of the AMSS model transmits a twisted martingale component to consumption, employment, and the tax rate.

4.6 Digression on Nonnegative Transfers

Throughout this lecture we have imposed that transfers $T_t = 0$.

AMSS [1] instead imposed a nonnegativity constraint $T_t \geq 0$ on transfers.

They also considered a special case of quasi-linear preferences, $u(c, l) = c + H(l)$.

In this case, $V_x(x, s) \leq 0$ is a non-positive martingale.

By the *martingale convergence theorem* $V_x(x, s)$ converges almost surely.

Furthermore, when the Markov chain $\Pi(s|s\_)$ and the government expenditure function $g(s)$ are such that g_t is perpetually random, $V_x(x, s)$ almost surely converges to zero.

For quasi-linear preferences, the first-order condition with respect to $n(s)$ becomes

$$(1 - \mu(s|s\_))(1 - u_l(s)) + \mu(s|s\_)n(s)u_{ll}(s) = 0$$

When $\mu(s|s\_) = \beta V_x(x(s), x)$ converges to zero, in the limit $u_l(s) = 1 = u_c(s)$, so that $\tau(x(s), s) = 0$.

Thus, in the limit, if g_t is perpetually random, the government accumulates sufficient assets to finance all expenditures from earnings on those assets, returning any excess revenues to the household as nonnegative lump sum transfers.

4.7 Code

The recursive formulation is implemented as follows

```
In [4]: using Dierckx


        mutable struct BellmanEquation_Recursive{TP <: Model, TI <: Integer, TR <:
  ↪Real}
            model::TP
```

```julia
    S::TI
    xbar::Array{TR}
    time_0::Bool
    z0::Array{Array}
    cFB::Vector{TR}
    nFB::Vector{TR}
    xFB::Vector{TR}
    zFB::Vector{Vector{TR}}
end


struct RecursiveAllocation{TP <: Model,
                           TI <: Integer,
                           TVg <: AbstractVector,
                           TT <: Tuple}
    model::TP
    mc::MarkovChain
    S::TI
    T::BellmanEquation_Recursive
    μgrid::TVg
    xgrid::TVg
    Vf::Array
    policies::TT
end


function RecursiveAllocation(model::Model, μgrid::AbstractArray)
    G = model.G
    S = size(model.Π, 1)             # number of states
    mc = MarkovChain(model.Π)
    # now find the first best allocation
    Vf, policies, T, xgrid = solve_time1_bellman(model, μgrid)
    T.time_0 = true                  # Bellman equation now solves time 0
↪problem
    return RecursiveAllocation(model, mc, S, T, μgrid, xgrid, Vf, policies)
end


function solve_time1_bellman(model::Model{TR}, μgrid::AbstractArray) where
↪{TR <: Real}
    Π = model.Π
    S = size(model.Π, 1)

    # First get initial fit from lucas stockey solution.
    # Need to change things to be ex_ante
    PP = SequentialAllocation(model)

    function incomplete_allocation(PP::SequentialAllocation,
                                   μ_::AbstractFloat,
                                   s_::Integer)
        c, n, x, V = time1_value(PP, μ_)
        return c, n, dot(Π[s_, :], x), dot(Π[s_, :], V)
    end

    cf = Array{Function}(undef, S, S)
    nf = Array{Function}(undef, S, S)
    xprimef = Array{Function}(undef, S, S)
    Vf = Vector{Function}(undef, S)
```

```julia
        xgrid = Array{TR}(undef, S, length(μgrid))

        for s_ in 1:S
            c = Array{TR}(undef, length(μgrid), S)
            n = Array{TR}(undef, length(μgrid), S)
            x = Array{TR}(undef, length(μgrid))
            V = Array{TR}(undef, length(μgrid))
            for (i_μ, μ) in enumerate(μgrid)
                c[i_μ, :], n[i_μ, :], x[i_μ], V[i_μ] =
                    incomplete_allocation(PP, μ, s_)
            end
            xprimes = repeat(x, 1, S)
            xgrid[s_, :] = x
            for sprime = 1:S
                splc = Spline1D(x[end:-1:1], c[:, sprime][end:-1:1], k=3)
                spln = Spline1D(x[end:-1:1], n[:, sprime][end:-1:1], k=3)
               splx = Spline1D(x[end:-1:1], xprimes[:, sprime][end:-1:1], k=3)
                cf[s_, sprime] = y -> splc(y)
                nf[s_, sprime] = y -> spln(y)
                xprimef[s_, sprime] = y -> splx(y)
              # cf[s_, sprime] = LinInterp(x[end:-1:1], c[:, sprime][end:-1:1])
              # nf[s_, sprime] = LinInterp(x[end:-1:1], n[:, sprime][end:-1:1])
                # xprimef[s_, sprime] = LinInterp(x[end:-1:1], xprimes[:,
↪sprime][end:-1:1])
            end
            splV = Spline1D(x[end:-1:1], V[end:-1:1], k=3)
            Vf[s_] = y -> splV(y)
            # Vf[s_] = LinInterp(x[end:-1:1], V[end:-1:1])
        end

        policies = [cf, nf, xprimef]

        # Create xgrid
        xbar = [maximum(minimum(xgrid)), minimum(maximum(xgrid))]
        xgrid = range(xbar[1], xbar[2], length = length(μgrid))

        # Now iterate on Bellman equation
        T = BellmanEquation_Recursive(model, xgrid, policies)
        diff = 1.0
        while diff > 1e-4
            PF = (i_x, x, s) -> get_policies_time1(T, i_x, x, s, Vf, xbar)
            Vfnew, policies = fit_policy_function(T, PF, xgrid)

            diff = 0.0
            for s=1:S
                diff = max(diff, maximum(abs, (Vf[s].(xgrid) - Vfnew[s].
↪(xgrid)) ./
                                                Vf[s].(xgrid)))
            end

            println("diff = $diff")
            Vf = copy(Vfnew)
        end

        return Vf, policies, T, xgrid
    end
```

```
function fit_policy_function(T::BellmanEquation_Recursive,
                             PF::Function,
                             xgrid::AbstractVector{TF}) where {TF <:
↪AbstractFloat}
    S = T.S
    # preallocation
    PFvec = Array{TF}(undef, 4S + 1, length(xgrid))
    cf = Array{Function}(undef, S, S)
    nf = Array{Function}(undef, S, S)
    xprimef = Array{Function}(undef, S, S)
    TTf = Array{Function}(undef, S, S)
    Vf = Vector{Function}(undef, S)
    # fit policy fuctions
    for s_ in 1:S
        for (i_x, x) in enumerate(xgrid)
            PFvec[:, i_x] = PF(i_x, x, s_)
        end
        splV = Spline1D(xgrid, PFvec[1,:], k=3)
        Vf[s_] = y -> splV(y)
        # Vf[s_] = LinInterp(xgrid, PFvec[1, :])
        for sprime=1:S
            splc = Spline1D(xgrid, PFvec[1 + sprime, :], k=3)
            spln = Spline1D(xgrid, PFvec[1 + S + sprime, :], k=3)
            splxprime = Spline1D(xgrid, PFvec[1 + 2S + sprime, :], k=3)
            splTT = Spline1D(xgrid, PFvec[1 + 3S + sprime, :], k=3)
            cf[s_, sprime] = y -> splc(y)
            nf[s_, sprime] = y -> spln(y)
            xprimef[s_, sprime] = y -> splxprime(y)
            TTf[s_, sprime] = y -> splTT(y)
        end
    end
    policies = (cf, nf, xprimef, TTf)
    return Vf, policies
end


function Tau(pab::RecursiveAllocation,
            c::AbstractArray,
        n::AbstractArray)
    model = pab.model
    Uc, Un = model.Uc(c, n), model.Un(c, n)
    return 1. .+ Un ./ (model.Θ .* Uc)
end

Tau(pab::RecursiveAllocation, c::Real, n::Real) = Tau(pab, [c], [n])


function time0_allocation(pab::RecursiveAllocation, B_::Real, s0::Integer)
    T, Vf = pab.T, pab.Vf
    xbar = T.xbar
    z0 = get_policies_time0(T, B_, s0, Vf, xbar)

    c0, n0, xprime0, T0 = z0[2], z0[3], z0[4], z0[5]
    return c0, n0, xprime0, T0
end


function simulate(pab::RecursiveAllocation,
                    B_::TF, s_0::Integer, T::Integer,
                   sHist::Vector=simulate(pab.mc, T, init=s_0)) where {TF <:
    AbstractFloat}
        model, mc, Vf, S = pab.model, pab.mc, pab.Vf, pab.S
        Π, Uc = model.Π, model.Uc
        cf, nf, xprimef, TTf = pab.policies

        cHist = Array{TF}(undef, T)
        nHist = Array{TF}(undef, T)
        Bhist = Array{TF}(undef, T)
        xHist = Array{TF}(undef, T)
        TauHist = Array{TF}(undef, T)
        THist = Array{TF}(undef, T)
        μHist = Array{TF}(undef, T)

        #time0
       cHist[1], nHist[1], xHist[1], THist[1]  = time0_allocation(pab, B_, s_0)
        TauHist[1] = Tau(pab, cHist[1], nHist[1])[s_0]
        Bhist[1] = B_
        μHist[1] = Vf[s_0](xHist[1])

        #time 1 onward
        for t in 2:T
            s_, x, s = sHist[t-1], xHist[t-1], sHist[t]
            c = Array{TF}(undef, S)
            n = Array{TF}(undef, S)
            xprime = Array{TF}(undef, S)
            TT = Array{TF}(undef, S)
            for sprime=1:S
                c[sprime], n[sprime], xprime[sprime], TT[sprime] =
                    cf[s_, sprime](x), nf[s_, sprime](x),
                    xprimef[s_, sprime](x), TTf[s_, sprime](x)
            end

            Tau_val = Tau(pab, c, n)[s]
            u_c = Uc(c, n)
            Eu_c = dot(Π[s_, :], u_c)

            μHist[t] = Vf[s](xprime[s])

            cHist[t], nHist[t], Bhist[t], TauHist[t] = c[s], n[s], x/Eu_c,
↪Tau_val
            xHist[t], THist[t] = xprime[s], TT[s]
        end
        return cHist, nHist, Bhist, xHist, TauHist, THist, μHist, sHist
    end


    function BellmanEquation_Recursive(model::Model{TF},
                                       xgrid::AbstractVector{TF},
                                  policies0::Array) where {TF <: AbstractFloat}

        S = size(model.Π, 1) # number of states
        xbar = [minimum(xgrid), maximum(xgrid)]
        time_0 = false
        z0 = Array{Array}(undef, length(xgrid), S)
        cf, nf, xprimef = policies0[1], policies0[2], policies0[3]
        for s in 1:S
```

```
            for (i_x, x) in enumerate(xgrid)
                cs = Array{TF}(undef, S)
                ns = Array{TF}(undef, S)
                xprimes = Array{TF}(undef, S)
                for j = 1:S
                    cs[j], ns[j], xprimes[j] = cf[s, j](x), nf[s, j](x),
↪xprimef[s, j](x)
                end
                z0[i_x, s] = vcat(cs, ns, xprimes, zeros(S))
            end
        end
        cFB, nFB, IFB, xFB, zFB = find_first_best(model, S, 2)
        return BellmanEquation_Recursive(model, S, xbar, time_0, z0, cFB, nFB,
↪xFB, zFB)
    end



    function get_policies_time1(T::BellmanEquation_Recursive,
                                i_x::Integer,
                                x::Real,
                                s_::Integer,
                                Vf::AbstractArray{Function},
                                xbar::AbstractVector)
        model, S = T.model, T.S
        β, Θ, G, Π = model.β, model.Θ, model.G, model.Π
        U,Uc,Un = model.U, model.Uc, model.Un

        S_possible = sum(Π[s_, :].>0)
        sprimei_possible = findall(Π[s_, :].>0)

        function objf(z, grad)
            c, xprime = z[1:S_possible], z[S_possible+1:2S_possible]
            n = (c .+ G[sprimei_possible]) ./ Θ[sprimei_possible]
            Vprime = [Vf[sprimei_possible[si]](xprime[si]) for si in 1:
↪S_possible]
            return -dot(Π[s_, sprimei_possible], U.(c, n) + β * Vprime)
        end

        function cons(out, z, grad)
            c, xprime, TT =
               z[1:S_possible], z[S_possible + 1:2S_possible], z[2S_possible +
    1:3S_possible]
            n = (c .+ G[sprimei_possible]) ./ Θ[sprimei_possible]
            u_c = Uc.(c, n)
            Eu_c = dot(Π[s_, sprimei_possible], u_c)
            out .= x * u_c/Eu_c - u_c .* (c - TT) - Un(c, n) .* n - β * xprime
        end
        function cons_no_trans(out, z, grad)
            c, xprime = z[1:S_possible], z[S_possible + 1:2S_possible]
            n = (c .+ G[sprimei_possible]) ./ Θ[sprimei_possible]
            u_c = Uc.(c, n)
            Eu_c = dot(Π[s_, sprimei_possible], u_c)
            out .= x * u_c / Eu_c - u_c .* c - Un(c, n) .* n - β * xprime
        end

        if model.transfers == true
```

```
                lb = vcat(zeros(S_possible), ones(S_possible)*xbar[1],
↪zeros(S_possible))
                if model.n_less_than_one == true
                    ub = vcat(ones(S_possible) - G[sprimei_possible],
                            ones(S_possible) * xbar[2], ones(S_possible))
                else
                    ub = vcat(100 * ones(S_possible),
                            ones(S_possible) * xbar[2],
                            100 * ones(S_possible))
                end
                init = vcat(T.z0[i_x, s_][sprimei_possible],
                            T.z0[i_x, s_][2S .+ sprimei_possible],
                            T.z0[i_x, s_][3S .+ sprimei_possible])
                opt = Opt(:LN_COBYLA, 3S_possible)
                equality_constraint!(opt, cons, zeros(S_possible))
            else
                lb = vcat(zeros(S_possible), ones(S_possible)*xbar[1])
                if model.n_less_than_one == true
                    ub = vcat(ones(S_possible)-G[sprimei_possible],
↪ones(S_possible)*xbar[2])
                else
                    ub = vcat(ones(S_possible), ones(S_possible) * xbar[2])
                end
                init = vcat(T.z0[i_x, s_][sprimei_possible],
                            T.z0[i_x, s_][2S .+ sprimei_possible])
                opt = Opt(:LN_COBYLA, 2S_possible)
                equality_constraint!(opt, cons_no_trans, zeros(S_possible))
            end
            init[init .> ub] = ub[init .> ub]
            init[init .< lb] = lb[init .< lb]

            min_objective!(opt, objf)
            lower_bounds!(opt, lb)
            upper_bounds!(opt, ub)
            maxeval!(opt, 10000000)
            maxtime!(opt, 10)
            ftol_rel!(opt, 1e-8)
            ftol_abs!(opt, 1e-8)

            (minf, minx, ret) = optimize(opt, init)

            if ret != :SUCCESS && ret != :ROUNDOFF_LIMITED && ret != :
↪MAXEVAL_REACHED &&
                ret != :FTOL_REACHED && ret != :MAXTIME_REACHED
                error("optimization failed: ret = $ret")
            end

            T.z0[i_x, s_][sprimei_possible] = minx[1:S_possible]
            T.z0[i_x, s_][S .+ sprimei_possible] = minx[1:S_possible] .+
↪G[sprimei_possible]
            T.z0[i_x, s_][2S .+ sprimei_possible] = minx[S_possible .+ 1:
↪2S_possible]
            if model.transfers == true
                T.z0[i_x, s_][3S .+ sprimei_possible] = minx[2S_possible + 1:
↪3S_possible]
            else
                T.z0[i_x, s_][3S .+ sprimei_possible] = zeros(S)
```

```
    end

    return vcat(-minf, T.z0[i_x, s_])
end


function get_policies_time0(T::BellmanEquation_Recursive,
                            B_::Real,
                            s0::Integer,
                            Vf::AbstractArray{Function},
                            xbar::AbstractVector)
    model = T.model
    β, Θ, G = model.β, model.Θ, model.G
    U, Uc, Un = model.U, model.Uc, model.Un

    function objf(z, grad)
        c, xprime = z[1], z[2]
        n = (c + G[s0]) / Θ[s0]
        return -(U(c, n) + β * Vf[s0](xprime))
    end

    function cons(z,grad)
        c, xprime, TT = z[1], z[2], z[3]
        n = (c + G[s0]) / Θ[s0]
        return -Uc(c, n) * (c - B_ - TT) - Un(c, n) * n - β * xprime
    end
    cons_no_trans(z, grad) = cons(vcat(z, 0), grad)

    if model.transfers == true
        lb = [0.0, xbar[1], 0.0]
        if model.n_less_than_one == true
            ub = [1 - G[s0], xbar[2], 100]
        else
            ub = [100.0, xbar[2], 100.0]
        end
        init = vcat(T.zFB[s0][1], T.zFB[s0][3], T.zFB[s0][4])
        init = [0.95124922, -1.15926816,  0.0]
        opt = Opt(:LN_COBYLA, 3)
        equality_constraint!(opt, cons)
    else
        lb = [0.0, xbar[1]]
        if model.n_less_than_one == true
            ub = [1-G[s0], xbar[2]]
        else
            ub = [100, xbar[2]]
        end
        init = vcat(T.zFB[s0][1], T.zFB[s0][3])
        init = [0.95124922, -1.15926816]
        opt = Opt(:LN_COBYLA, 2)
        equality_constraint!(opt, cons_no_trans)
    end
    init[init .> ub] = ub[init .> ub]
    init[init .< lb] = lb[init .< lb]


    min_objective!(opt, objf)
    lower_bounds!(opt, lb)
    upper_bounds!(opt, ub)
```

```
        maxeval!(opt, 100000000)
        maxtime!(opt, 30)

        (minf, minx, ret) = optimize(opt, init)

        if ret != :SUCCESS && ret != :ROUNDOFF_LIMITED && ret != :
↪MAXEVAL_REACHED &&
            ret != :FTOL_REACHED
                error("optimization failed: ret = $ret")
        end

        if model.transfers == true
            return -minf, minx[1], minx[1]+G[s0], minx[2], minx[3]
        else
            return -minf, minx[1], minx[1]+G[s0], minx[2], 0
        end
    end
```

```
Out[4]: get_policies_time0 (generic function with 2 methods)
```

5 Examples

We now turn to some examples.

5.1 Anticipated One-Period War

In our lecture on optimal taxation with state contingent debt we studied how the government manages uncertainty in a simple setting.

As in that lecture, we assume the one-period utility function

$$u(c,n) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{n^{1+\gamma}}{1+\gamma}$$

Note

For convenience in matching our computer code, we have expressed utility as a function of n rather than leisure l

We consider the same government expenditure process studied in the lecture on optimal taxation with state contingent debt.

Government expenditures are known for sure in all periods except one

- For $t < 3$ or $t > 3$ we assume that $g_t = g_l = 0.1$.
- At $t = 3$ a war occurs with probability 0.5.
  - If there is war, $g_3 = g_h = 0.2$.
  - If there is no war $g_3 = g_l = 0.1$.

A useful trick is to define components of the state vector as the following six (t, g) pairs:

$$(0, g_l), (1, g_l), (2, g_l), (3, g_l), (3, g_h), (t \geq 4, g_l)$$

We think of these 6 states as corresponding to $s = 1, 2, 3, 4, 5, 6$.

The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The government expenditure at each state is

$$g = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.2 \\ 0.1 \end{pmatrix}$$

We assume the same utility parameters as in the Lucas-Stokey economy.

This utility function is implemented in the following constructor

```
In [5]: function crra_utility(;
            β = 0.9,
            σ = 2.0,
            γ = 2.0,
            Π = 0.5 * ones(2, 2),
            G = [0.1, 0.2],
            Θ = ones(Float64, 2),
            transfers = false
            )
            function U(c, n)
                if σ == 1.0
                    U = log(c)
                else
                    U = (c.^(1.0 - σ) - 1.0) / (1.0 - σ)
                end
                return U - n.^(1 + γ) / (1 + γ)
            end
            # Derivatives of utility function
            Uc(c,n) =  c.^(-σ)
            Ucc(c,n) = -σ * c.^(-σ - 1.0)
            Un(c,n) = -n.^γ
            Unn(c,n) = -γ * n.^(γ - 1.0)
            n_less_than_one = false
            return Model(β, Π, G, Θ, transfers,
                            U, Uc, Ucc, Un, Unn, n_less_than_one)
        end
```

```
Out[5]: crra_utility (generic function with 1 method)
```

The following figure plots the Ramsey plan under both complete and incomplete markets for both possible realizations of the state at time $t = 3$.

Optimal policies when the government has access to state contingent debt are represented by black lines, while the optimal policies when there is only a risk free bond are in red.

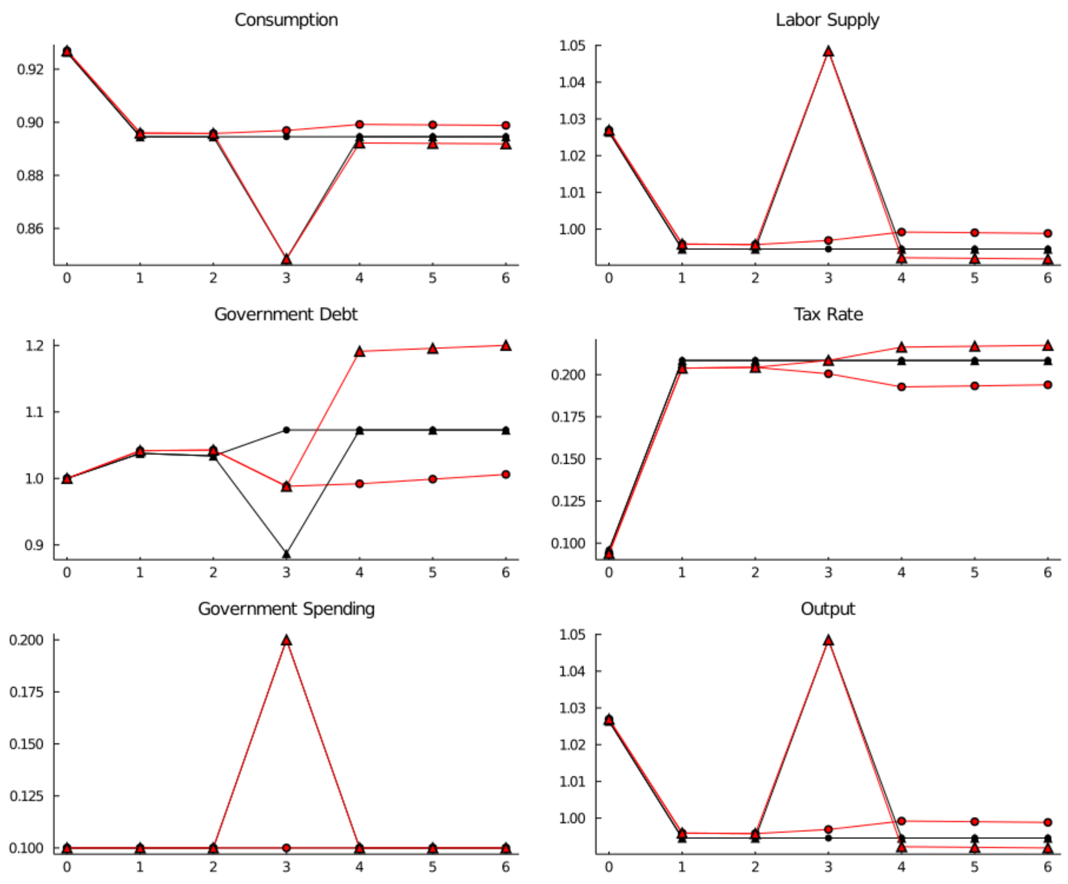
Paths with circles are histories in which there is peace, while those with triangle denote war.

```
In [6]: time_example = crra_utility(G=[0.1, 0.1, 0.1, 0.2, 0.1, 0.1],
                                    Θ = ones(6)) # Θ can in principle be random

        time_example.Π = [ 0.0 1.0 0.0 0.0 0.0 0.0;
                           0.0 0.0 1.0 0.0 0.0 0.0;
                           0.0 0.0 0.0 0.5 0.5 0.0;
                           0.0 0.0 0.0 0.0 0.0 1.0;
                           0.0 0.0 0.0 0.0 0.0 1.0;
                           0.0 0.0 0.0 0.0 0.0 1.0]

        # Initialize μgrid for value function iteration
        μgrid = range(-0.7, 0.01, length = 200)

        time_example.transfers = true  # Government can use transfers
        time_sequential = SequentialAllocation(time_example) # Solve sequential
  ↪problem

        time_bellman = RecursiveAllocation(time_example, μgrid)

        sHist_h = [1, 2, 3, 4, 6, 6, 6]
        sHist_l = [1, 2, 3, 5, 6, 6, 6]

        sim_seq_h = simulate(time_sequential, 1., 1, 7, sHist_h)
        sim_bel_h = simulate(time_bellman, 1., 1, 7, sHist_h)
        sim_seq_l = simulate(time_sequential, 1., 1, 7, sHist_l)
        sim_bel_l = simulate(time_bellman, 1., 1, 7, sHist_l)

        using Plots
        gr(fmt=:png);
        titles = hcat("Consumption", "Labor Supply", "Government Debt",
                      "Tax Rate", "Government Spending", "Output")
        sim_seq_l_plot = hcat(sim_seq_l[1:3]..., sim_seq_l[4],
                              time_example.G[sHist_l],
                              time_example.Θ[sHist_l] .* sim_seq_l[2])
        sim_bel_l_plot = hcat(sim_bel_l[1:3]..., sim_bel_l[5],
                              time_example.G[sHist_l],
                              time_example.Θ[sHist_l] .* sim_bel_l[2])
        sim_seq_h_plot = hcat(sim_seq_h[1:3]..., sim_seq_h[4],
                              time_example.G[sHist_h],
                              time_example.Θ[sHist_h] .* sim_seq_h[2])
        sim_bel_h_plot = hcat(sim_bel_h[1:3]..., sim_bel_h[5],
                              time_example.G[sHist_h],
                              time_example.Θ[sHist_h] .* sim_bel_h[2])
        p = plot(size = (920, 750), layout =(3, 2),
         xaxis=(0:6), grid=false, titlefont=Plots.font("sans-serif", 10))
        plot!(p, title = titles)
        for i=1:6
            plot!(p[i], 0:6, sim_seq_l_plot[:, i], marker=:circle, color=:black,
  ↪lab="")
            plot!(p[i], 0:6, sim_bel_l_plot[:, i], marker=:circle, color=:red,
  ↪lab="")
            plot!(p[i], 0:6, sim_seq_h_plot[:, i], marker=:utriangle, color=:
  ↪black, lab="")
```

```
        plot!(p[i], 0:6, sim_bel_h_plot[:, i], marker=:utriangle, color=:red,
↪lab="")
    end
    p
```

```
    diff = 0.019541811391361420
diff = 0.020296606343314445
diff = 0.01825290333882874
diff = 0.01732330130265356
diff = 0.002438290769982288
diff = 0.0017049947758874273
diff = 0.001491200081454862
diff = 0.0008415716013777662
diff = 0.0006551802043226648
diff = 0.0005101110034210979
diff = 0.00045899123671075637
diff = 0.0004130378473633363
diff = 0.00037165554806329687
diff = 0.00033441325852100494
diff = 0.0003008983054583036
diff = 0.0002707536359176724
diff = 0.00024362702609734803
diff = 0.0002192208026612293
diff = 0.00019723701023064875
diff = 0.00017750553015148367
diff = 0.00015972217622243714
diff = 0.00014373496579809245
diff = 0.00012934400797986716
diff = 0.00011639475826865351
diff = 0.00010474335075915266
diff = 9.425940404825291e-5
```

Out[6]:



How a Ramsey planner responds to war depends on the structure of the asset market.

If it is able to trade state-contingent debt, then at time $t = 2$

- the government purchases an Arrow security that pays off when $g_3 = g_h$
- the government sells an Arrow security that pays off when $g_3 = g_l$
- These purchases are designed in such a way that regardless of whether or not there is a war at $t = 3$, the government will begin period $t = 4$ with the *same* government debt.

This pattern facilities smoothing tax rates across states.

The government without state contingent debt cannot do this.

Instead, it must enter time $t = 3$ with the same level of debt falling due whether there is peace or war at $t = 3$.

It responds to this constraint by smoothing tax rates across time.

To finance a war it raises taxes and issues more debt.

To service the additional debt burden, it raises taxes in all future periods.

The absence of state contingent debt leads to an important difference in the optimal tax policy.

When the Ramsey planner has access to state contingent debt, the optimal tax policy is history independent

- the tax rate is a function of the current level of government spending only, given the Lagrange multiplier on the implementability constraint.

Without state contingent debt, the optimal tax rate is history dependent.

- A war at time $t = 3$ causes a permanent increase in the tax rate.

5.1.1 Perpetual War Alert

History dependence occurs more dramatically in a case in which the government perpetually faces the prospect of war.

This case was studied in the final example of the lecture on optimal taxation with state-contingent debt.

There, each period the government faces a constant probability, 0.5, of war.

In addition, this example features the following preferences

$$u(c, n) = \log(c) + 0.69 \log(1 - n)$$

In accordance, we will re-define our utility function

```
In [7]: function log_utility(;β = 0.9,
                              ψ = 0.69,
                              Π = 0.5 * ones(2, 2),
                              G = [0.1, 0.2],
                              Θ = ones(2),
                              transfers = false)
            # Derivatives of utility function
            U(c,n) = log(c) + ψ * log(1 - n)
            Uc(c,n) = 1 ./ c
            Ucc(c,n) = -c.^(-2.0)
            Un(c,n) = -ψ ./ (1.0 .- n)
            Unn(c,n) = -ψ ./ (1.0 .- n).^2.0
            n_less_than_one = true
            return Model(β, Π, G, Θ, transfers,
                            U, Uc, Ucc, Un, Unn, n_less_than_one)
        end
```

```
Out[7]: log_utility (generic function with 1 method)
```

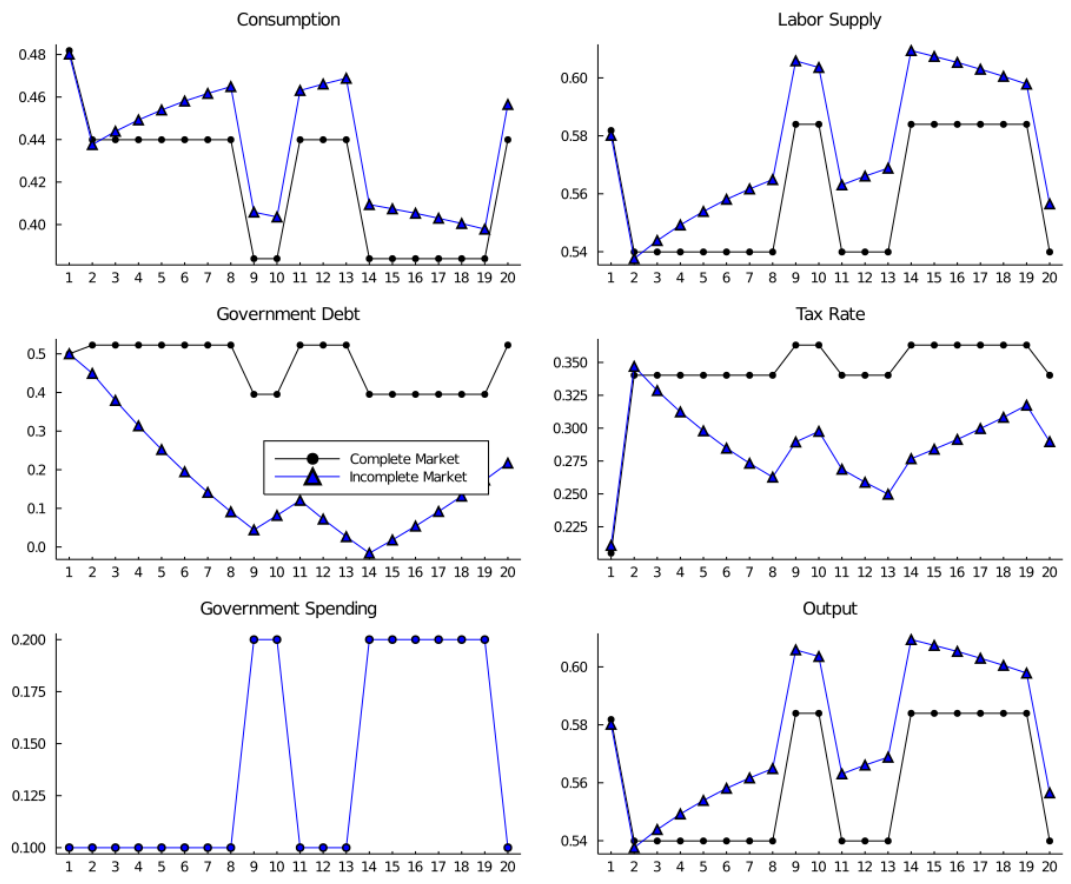
With these preferences, Ramsey tax rates will vary even in the Lucas-Stokey model with state-contingent debt.

The figure below plots optimal tax policies for both the economy with state contingent debt (circles) and the economy with only a risk-free bond (triangles)

```
In [8]: log_example = log_utility()

        log_example.transfers = true                             # Government can␣
  ↪use transfers
        log_sequential = SequentialAllocation(log_example)       # Solve␣
  ↪sequential problem
        log_bellman = RecursiveAllocation(log_example, μgrid)    # Solve recursive␣
  ↪problem

        T = 20
        sHist = [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 2, 2, 2, 2, 2, 2, 1]

        #simulate
        sim_seq = simulate(log_sequential, 0.5, 1, T, sHist)
        sim_bel = simulate(log_bellman, 0.5, 1, T, sHist)

        sim_seq_plot = hcat(sim_seq[1:3]...,
                    sim_seq[4], log_example.G[sHist], log_example.Θ[sHist] .*
↪sim_seq[2])
        sim_bel_plot = hcat(sim_bel[1:3]...,
                    sim_bel[5], log_example.G[sHist], log_example.Θ[sHist] .*
↪sim_bel[2])

        #plot policies
        p = plot(size = (920, 750), layout = grid(3, 2),
                 xaxis=(0:T), grid=false, titlefont=Plots.font("sans-serif", 10))
        labels = fill(("", ""), 6)
        labels[3] = ("Complete Market", "Incomplete Market")
        plot!(p, title = titles)
        for i = vcat(collect(1:4), 6)
            plot!(p[i], sim_seq_plot[:, i], marker=:circle, color=:black,
↪lab=labels[i][1])
            plot!(p[i], sim_bel_plot[:, i], marker=:utriangle, color=:blue,
↪lab=labels[i][2],
                  legend=:bottomright)
        end
        plot!(p[5], sim_seq_plot[:, 5], marker=:circle, color=:blue, lab="")
```

```
    diff = 0.0007972378476372139
diff = 0.0006423560333504441
diff = 0.0005517441622530832
diff = 0.0004855393001335185
diff = 0.0004226590836939342
diff = 0.0003755067231697640
diff = 0.0003294032122270672
diff = 0.0002933723232171897
diff = 0.0002585679504824062
diff = 0.0002304262486527987
diff = 0.0002031214087191915
diff = 0.0001811528283364364
diff = 0.0001603437475197024
diff = 0.0001429496057340243
diff = 0.0001267581715890033
diff = 0.0001129520548991428
diff = 0.0001003087897706202
diff = 8.934103186095062e-5
```

Out[8]:



When the government experiences a prolonged period of peace, it is able to reduce government debt and set permanently lower tax rates.

However, the government finances a long war by borrowing and raising taxes.

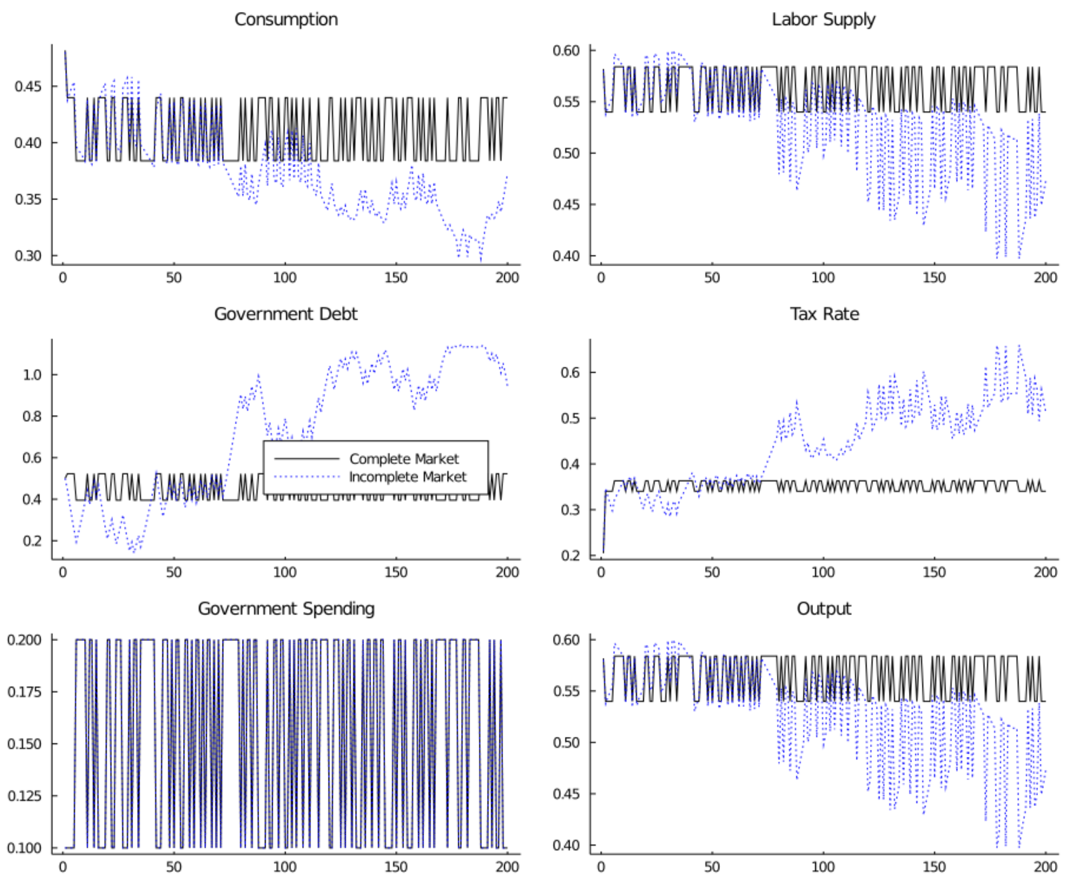
This results in a drift away from policies with state contingent debt that depends on the history of shocks.

This is even more evident in the following figure that plots the evolution of the two policies over 200 periods

```
In [9]: T_long = 200
        sim_seq_long = simulate(log_sequential, 0.5, 1, T_long)
        sHist_long = sim_seq_long[end-2]
        sim_bel_long = simulate(log_bellman, 0.5, 1, T_long, sHist_long)
        sim_seq_long_plot = hcat(sim_seq_long[1:4]...,
                     log_example.G[sHist_long], log_example.Θ[sHist_long] .*
 ↪sim_seq_long[2])
        sim_bel_long_plot = hcat(sim_bel_long[1:3]..., sim_bel_long[5],
                     log_example.G[sHist_long], log_example.Θ[sHist_long] .*
 ↪sim_bel_long[2])

        p = plot(size = (920, 750), layout = (3, 2), xaxis=(0:50:T_long),
 ↪grid=false,
                 titlefont=Plots.font("sans-serif", 10))
        plot!(p, title = titles)
        for i = 1:6
            plot!(p[i], sim_seq_long_plot[:, i], color=:black, linestyle=:solid,
        lab=labels[i][1])
```

```
            plot!(p[i], sim_bel_long_plot[:, i], color=:blue, linestyle=:dot,
↪lab=labels[i][2],
                  legend=:bottomright)
        end
        p
```

Out[9]:



Footnotes

[**1**] In an allocation that solves the Ramsey problem and that levies distorting taxes on labor, why would the government ever want to hand revenues back to the private sector? It would not in an economy with state-contingent debt, since any such allocation could be improved by lowering distortionary taxes rather than handing out lump-sum transfers. But without state-contingent debt there can be circumstances when a government would like to make lump-sum transfers to the private sector.

[**2**] From the first-order conditions for the Ramsey problem, there exists another realization $\tilde{s}^t$ with the same history up until the previous period, i.e., $\tilde{s}^{t-1} = s^{t-1}$, but where the multiplier on constraint (11) takes a positive value, so $\gamma_t(\tilde{s}^t) > 0$.

References

[1] S. Rao Aiyagari, Albert Marcet, Thomas J. Sargent, and Juha Seppala. Optimal Taxation without State-Contingent Debt. *Journal of Political Economy*, 110(6):1220–1254, December 2002.

[2] Robert J Barro. On the Determination of the Public Debt. *Journal of Political Economy*, 87(5):940–971, 1979.

[3] Robert E Lucas, Jr. and Nancy L Stokey. Optimal Fiscal and Monetary Policy in an Economy without Capital. *Journal of monetary Economics*, 12(3):55–93, 1983.