

Optimal Savings I: The Permanent Income Model

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2 Overview

This lecture describes a rational expectations version of the famous permanent income model of Milton Friedman [\[5\]](#).

Robert Hall cast Friedman's model within a linear-quadratic setting [\[6\]](#).

Like Hall, we formulate an infinite-horizon linear-quadratic savings problem.

We use the model as a vehicle for illustrating

- alternative formulations of the *state* of a dynamic system
- the idea of *cointegration*
- impulse response functions
- the idea that changes in consumption are useful as predictors of movements in income

Background readings on the linear-quadratic-Gaussian permanent income model are Hall's [\[6\]](#) and chapter 2 of [\[11\]](#).

2.1 Setup

```
In [1]: using InstantiateFromURL
        # optionally add arguments to force installation: instantiate = true,
        ↪precompile = true
        github_project("QuantEcon/quantecon-notebooks-julia", version = "0.8.0")
```

```
In [2]: using LinearAlgebra, Statistics
```

3 The Savings Problem

In this section we state and solve the savings and consumption problem faced by the consumer.

3.1 Preliminaries

We use a class of stochastic processes called [martingales](#).

A discrete time martingale is a stochastic process (i.e., a sequence of random variables) $\{X_t\}$ with finite mean at each t and satisfying

$$\mathbb{E}_t[X_{t+1}] = X_t, \quad t = 0, 1, 2, \dots$$

Here $\mathbb{E}_t := \mathbb{E}[\cdot | \mathcal{F}_t]$ is a conditional mathematical expectation conditional on the time t *information set* \mathcal{F}_t .

The latter is just a collection of random variables that the modeler declares to be visible at t .

- When not explicitly defined, it is usually understood that $\mathcal{F}_t = \{X_t, X_{t-1}, \dots, X_0\}$.

Martingales have the feature that the history of past outcomes provides no predictive power for changes between current and future outcomes.

For example, the current wealth of a gambler engaged in a “fair game” has this property.

One common class of martingales is the family of *random walks*.

A **random walk** is a stochastic process $\{X_t\}$ that satisfies

$$X_{t+1} = X_t + w_{t+1}$$

for some iid zero mean *innovation* sequence $\{w_t\}$.

Evidently X_t can also be expressed as

$$X_t = \sum_{j=1}^t w_j + X_0$$

Not every martingale arises as a random walk (see, for example, [Wald’s martingale](#)).

3.2 The Decision Problem

A consumer has preferences over consumption streams that are ordered by the utility functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{1}$$

where

- \mathbb{E}_t is the mathematical expectation conditioned on the consumer’s time t information

- c_t is time t consumption
- u is a strictly concave one-period utility function
- $\beta \in (0, 1)$ is a discount factor

The consumer maximizes (1) by choosing a consumption, borrowing plan $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$c_t + b_t = \frac{1}{1+r} b_{t+1} + y_t \quad t \geq 0 \quad (2)$$

Here

- y_t is an exogenous endowment process
- $r > 0$ is a time-invariant risk-free net interest rate
- b_t is one-period risk-free debt maturing at t

The consumer also faces initial conditions b_0 and y_0 , which can be fixed or random.

3.3 Assumptions

For the remainder of this lecture, we follow Friedman and Hall in assuming that $(1+r)^{-1} = \beta$. Regarding the endowment process, we assume it has the [state-space representation](#).

$$\begin{aligned} z_{t+1} &= Az_t + Cw_{t+1} \\ y_t &= Uz_t \end{aligned} \quad (3)$$

where

- $\{w_t\}$ is an iid vector process with $\mathbb{E}w_t = 0$ and $\mathbb{E}w_t w_t' = I$
- the [spectral radius](#) of A satisfies $\rho(A) < \sqrt{1/\beta}$
- U is a selection vector that pins down y_t as a particular linear combination of components of z_t .

The restriction on $\rho(A)$ prevents income from growing so fast that discounted geometric sums of some quadratic forms to be described below become infinite.

Regarding preferences, we assume the quadratic utility function

$$u(c_t) = -(c_t - \gamma)^2$$

where γ is a bliss level of consumption

Note

Along with this quadratic utility specification, we allow consumption to be negative. However, by choosing parameters appropriately, we can make the probability that the model generates negative consumption paths over finite time horizons as low as desired.

Finally, we impose the *no Ponzi scheme* condition

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t b_t^2 \right] < \infty \quad (4)$$

This condition rules out an always-borrow scheme that would allow the consumer to enjoy bliss consumption forever.

3.4 First-Order Conditions

First-order conditions for maximizing (1) subject to (2) are

$$\mathbb{E}_t[u'(c_{t+1})] = u'(c_t), \quad t = 0, 1, \dots \quad (5)$$

These optimality conditions are also known as *Euler equations*.

If you're not sure where they come from, you can find a proof sketch in the [appendix](#).

With our quadratic preference specification, (5) has the striking implication that consumption follows a martingale:

$$\mathbb{E}_t[c_{t+1}] = c_t \quad (6)$$

(In fact quadratic preferences are *necessary* for this conclusion Section ??)

One way to interpret (6) is that consumption will change only when “new information” about permanent income is revealed.

These ideas will be clarified below.

3.5 The Optimal Decision Rule

Now let's deduce the optimal decision rule Section ??.

Note

One way to solve the consumer's problem is to apply *dynamic programming* as in [this lecture](#). We do this later. But first we use an alternative approach that is revealing and shows the work that dynamic programming does for us behind the scenes.

In doing so, we need to combine

1. the optimality condition (6)
2. the period-by-period budget constraint (2), and
3. the boundary condition (4)

To accomplish this, observe first that (4) implies $\lim_{t \rightarrow \infty} \beta^{\frac{t}{2}} b_{t+1} = 0$.

Using this restriction on the debt path and solving (2) forward yields

$$b_t = \sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) \quad (7)$$

Take conditional expectations on both sides of (7) and use the martingale property of consumption and the *law of iterated expectations* to deduce

$$b_t = \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}] - \frac{c_t}{1-\beta} \quad (8)$$

Expressed in terms of c_t we get

$$c_t = (1-\beta) \left[\sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}] - b_t \right] = \frac{r}{1+r} \left[\sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}] - b_t \right] \quad (9)$$

where the last equality uses $(1+r)\beta = 1$.

These last two equations assert that consumption equals *economic income*

- **financial wealth** equals $-b_t$
- **non-financial wealth** equals $\sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}]$
- **total wealth** equals the sum of financial and non-financial wealth
- **A marginal propensity to consume out of total wealth** equals the interest factor $\frac{r}{1+r}$
- **economic income** equals
 - a constant marginal propensity to consume times the sum of non-financial wealth and financial wealth
 - the amount the consumer can consume while leaving its wealth intact

3.5.1 Responding to the State

The *state* vector confronting the consumer at t is $[b_t \quad z_t]$.

Here

- z_t is an *exogenous* component, unaffected by consumer behavior
- b_t is an *endogenous* component (since it depends on the decision rule)

Note that z_t contains all variables useful for forecasting the consumer's future endowment.

It is plausible that current decisions c_t and b_{t+1} should be expressible as functions of z_t and b_t .

This is indeed the case.

In fact, from [this discussion](#) we see that

$$\sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}] = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] = U(I - \beta A)^{-1} z_t$$

Combining this with (9) gives

$$c_t = \frac{r}{1+r} [U(I - \beta A)^{-1} z_t - b_t] \quad (10)$$

Using this equality to eliminate c_t in the budget constraint (2) gives

$$\begin{aligned} b_{t+1} &= (1+r)(b_t + c_t - y_t) \\ &= (1+r)b_t + r[U(I - \beta A)^{-1} z_t - b_t] - (1+r)y_t \\ &= b_t + U[r(I - \beta A)^{-1} - (1+r)I]z_t \\ &= b_t + U(I - \beta A)^{-1}(A - I)z_t \end{aligned}$$

To get from the second last to the last expression in this chain of equalities is not trivial.

A key is to use the fact that $(1+r)\beta = 1$ and $(I - \beta A)^{-1} = \sum_{j=0}^{\infty} \beta^j A^j$.

We've now successfully written c_t and b_{t+1} as functions of b_t and z_t .

3.5.2 A State-Space Representation

We can summarize our dynamics in the form of a linear state-space system governing consumption, debt and income:

$$\begin{aligned} z_{t+1} &= Az_t + Cw_{t+1} \\ b_{t+1} &= b_t + U[(I - \beta A)^{-1}(A - I)]z_t \\ y_t &= Uz_t \\ c_t &= (1 - \beta)[U(I - \beta A)^{-1}z_t - b_t] \end{aligned} \tag{11}$$

To write this more succinctly, let

$$x_t = \begin{bmatrix} z_t \\ b_t \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ U(I - \beta A)^{-1}(A - I) & 1 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

and

$$\tilde{U} = \begin{bmatrix} U & 0 \\ (1 - \beta)U(I - \beta A)^{-1} & -(1 - \beta) \end{bmatrix}, \quad \tilde{y}_t = \begin{bmatrix} y_t \\ c_t \end{bmatrix}$$

Then we can express equation (11) as

$$\begin{aligned} x_{t+1} &= \tilde{A}x_t + \tilde{C}w_{t+1} \\ \tilde{y}_t &= \tilde{U}x_t \end{aligned} \tag{12}$$

We can use the following formulas from [linear state space models](#) to compute population mean $\mu_t = \mathbb{E}x_t$ and covariance $\Sigma_t := \mathbb{E}[(x_t - \mu_t)(x_t - \mu_t)']$

$$\mu_{t+1} = \tilde{A}\mu_t \quad \text{with } \mu_0 \text{ given} \tag{13}$$

$$\Sigma_{t+1} = \tilde{A}\Sigma_t\tilde{A}' + \tilde{C}\tilde{C}' \quad \text{with } \Sigma_0 \text{ given} \tag{14}$$

We can then compute the mean and covariance of \tilde{y}_t from

$$\begin{aligned} \mu_{y,t} &= \tilde{U}\mu_t \\ \Sigma_{y,t} &= \tilde{U}\Sigma_t\tilde{U}' \end{aligned} \tag{15}$$

3.5.3 A Simple Example with iid Income

To gain some preliminary intuition on the implications of (11), let's look at a highly stylized example where income is just iid.

(Later examples will investigate more realistic income streams)

In particular, let $\{w_t\}_{t=1}^{\infty}$ be iid and scalar standard normal, and let

$$z_t = \begin{bmatrix} z_t^1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = [1 \quad \mu], \quad C = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$$

Finally, let $b_0 = z_0^1 = 0$.

Under these assumptions we have $y_t = \mu + \sigma w_t \sim N(\mu, \sigma^2)$.

Further, if you work through the state space representation, you will see that

$$b_t = -\sigma \sum_{j=1}^{t-1} w_j$$
$$c_t = \mu + (1 - \beta)\sigma \sum_{j=1}^t w_j$$

Thus income is iid and debt and consumption are both Gaussian random walks.

Defining assets as $-b_t$, we see that assets are just the cumulative sum of unanticipated incomes prior to the present date.

The next figure shows a typical realization with $r = 0.05$, $\mu = 1$, and $\sigma = 0.15$

```
In [3]: using Plots, Random
gr(fmt=:png);

Random.seed!(42)

const r = 0.05
const β = 1.0 / (1.0 + r)
const T = 60
const σ = 0.15
const μ = 1.0

function time_path2()
    w = randn(T+1)
    w[1] = 0.0
    b = zeros(T+1)
    for t=2:T+1
        b[t] = sum(w[1:t])
    end
    b .*= -σ
    c = μ .+ (1.0 - β) .* (σ .* w .- b)
    return w, b, c
end

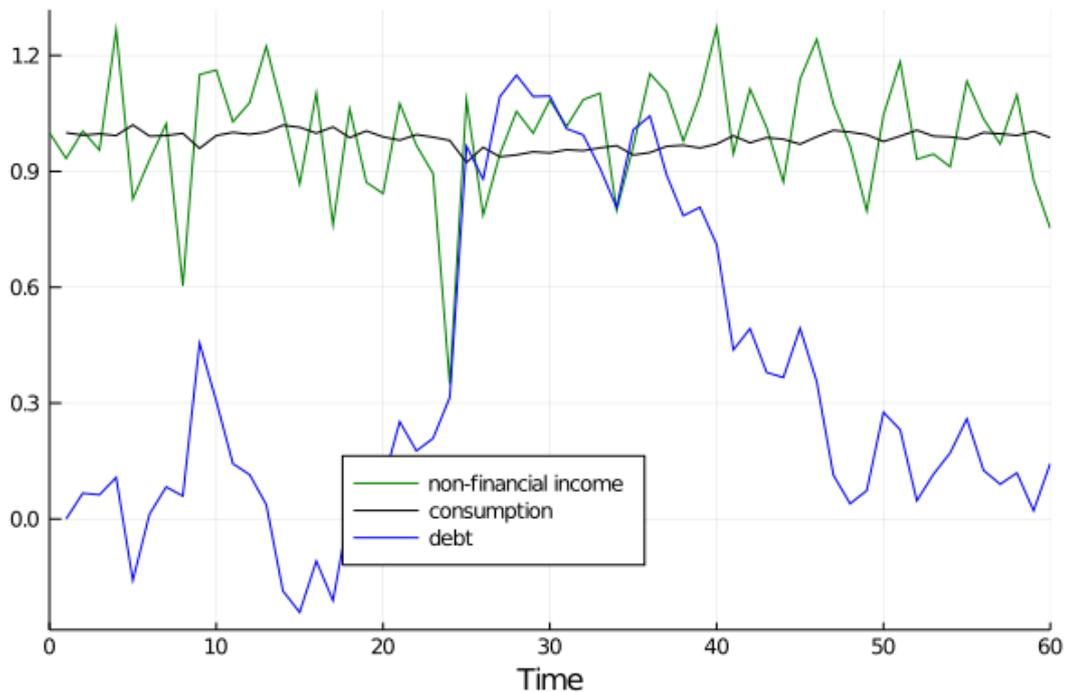
w, b, c = time_path2()
p = plot(0:T, μ .+ σ .* w, color = :green, label = "non-financial income")
```

```

plot!(c, color = :black, label = "consumption")
plot!(b, color = :blue, label = "debt")
plot!(xlabel = "Time", linewidth = 2, alpha = 0.7,
      xlims = (0, T), legend = :bottom)

```

Out[3]:



Observe that consumption is considerably smoother than income.

The figure below shows the consumption paths of 250 consumers with independent income streams

```

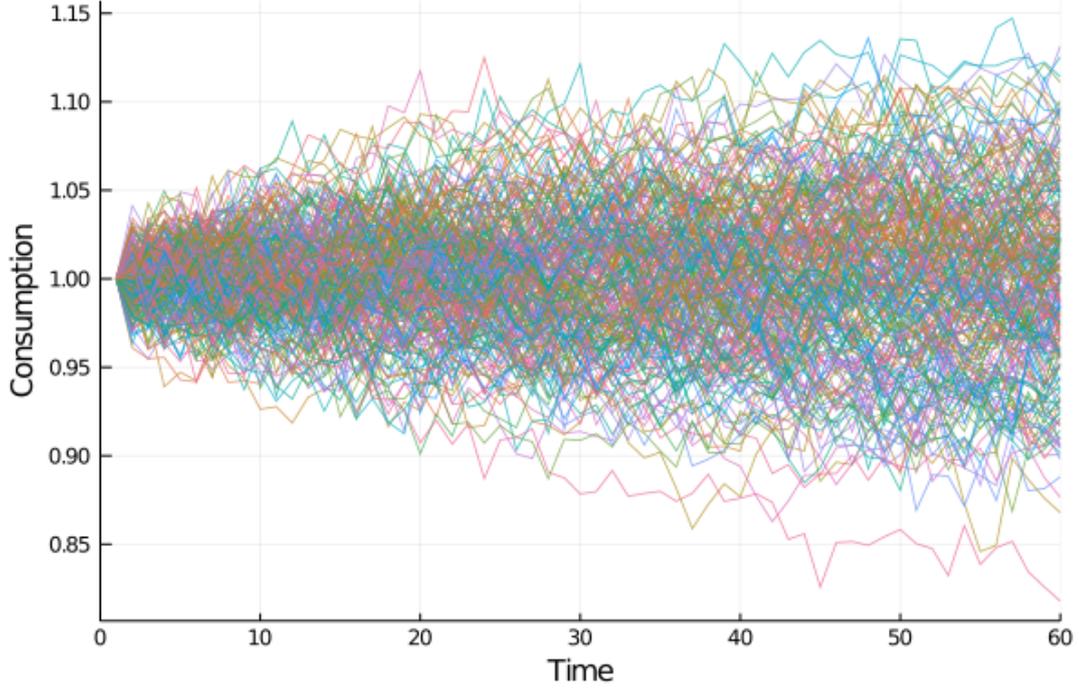
In [4]: time_paths = []
n = 250

for i in 1:n
    push!(time_paths, time_path2()[3])
end

p = plot(time_paths, linewidth = 0.8, alpha=0.7, legend = :none)
plot!(xlabel = "Time", ylabel = "Consumption", xlims = (0, T))

```

Out[4]:



4 Alternative Representations

In this section we shed more light on the evolution of savings, debt and consumption by representing their dynamics in several different ways.

4.1 Hall's Representation

Hall [6] suggested an insightful way to summarize the implications of LQ permanent income theory.

First, to represent the solution for b_t , shift (9) forward one period and eliminate b_{t+1} by using (2) to obtain

$$c_{t+1} = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \mathbb{E}_{t+1}[y_{t+j+1}] - (1 - \beta) [\beta^{-1}(c_t + b_t - y_t)]$$

If we add and subtract $\beta^{-1}(1 - \beta) \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t y_{t+j}$ from the right side of the preceding equation and rearrange, we obtain

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \{ \mathbb{E}_{t+1}[y_{t+j+1}] - \mathbb{E}_t[y_{t+j+1}] \} \quad (16)$$

The right side is the time $t + 1$ *innovation to the expected present value* of the endowment process $\{y_t\}$.

We can represent the optimal decision rule for (c_t, b_{t+1}) in the form of (16) and (8), which we repeat:

$$b_t = \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[y_{t+j}] - \frac{1}{1-\beta} c_t \quad (17)$$

Equation (17) asserts that the consumer's debt due at t equals the expected present value of its endowment minus the expected present value of its consumption stream.

A high debt thus indicates a large expected present value of surpluses $y_t - c_t$.

Recalling again our discussion on [forecasting geometric sums](#), we have

$$\begin{aligned} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j} &= U(I - \beta A)^{-1} z_t \\ \mathbb{E}_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} &= U(I - \beta A)^{-1} z_{t+1} \\ \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j+1} &= U(I - \beta A)^{-1} A z_t \end{aligned}$$

Using these formulas together with (3) and substituting into (16) and (17) gives the following representation for the consumer's optimum decision rule:

$$\begin{aligned} c_{t+1} &= c_t + (1 - \beta)U(I - \beta A)^{-1} C w_{t+1} \\ b_t &= U(I - \beta A)^{-1} z_t - \frac{1}{1 - \beta} c_t \\ y_t &= U z_t \\ z_{t+1} &= A z_t + C w_{t+1} \end{aligned} \quad (18)$$

Representation (18) makes clear that

- The state can be taken as (c_t, z_t) .
 - The endogenous part is c_t and the exogenous part is z_t .
 - Debt b_t has disappeared as a component of the state because it is encoded in c_t .
- Consumption is a random walk with innovation $(1 - \beta)U(I - \beta A)^{-1} C w_{t+1}$.
 - This is a more explicit representation of the martingale result in (6).

4.2 Cointegration

Representation (18) reveals that the joint process $\{c_t, b_t\}$ possesses the property that Engle and Granger [4] called [cointegration](#).

Cointegration is a tool that allows us to apply powerful results from the theory of stationary stochastic processes to (certain transformations of) nonstationary models.

To apply cointegration in the present context, suppose that z_t is asymptotically stationary Section ??.

Despite this, both c_t and b_t will be non-stationary because they have unit roots (see (11) for b_t).

Nevertheless, there is a linear combination of c_t, b_t that *is* asymptotically stationary.

In particular, from the second equality in (18) we have

$$(1 - \beta)b_t + c_t = (1 - \beta)U(I - \beta A)^{-1}z_t \quad (19)$$

Hence the linear combination $(1 - \beta)b_t + c_t$ is asymptotically stationary.

Accordingly, Granger and Engle would call $[(1 - \beta) \ 1]$ a **cointegrating vector** for the state.

When applied to the nonstationary vector process $[b_t \ c_t]'$, it yields a process that is asymptotically stationary.

Equation (19) can be rearranged to take the form

$$(1 - \beta)b_t + c_t = (1 - \beta)\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j}. \quad (20)$$

Equation (20) asserts that the *cointegrating residual* on the left side equals the conditional expectation of the geometric sum of future incomes on the right Section ??.

4.3 Cross-Sectional Implications

Consider again (18), this time in light of our discussion of distribution dynamics in the [lecture on linear systems](#).

The dynamics of c_t are given by

$$c_{t+1} = c_t + (1 - \beta)U(I - \beta A)^{-1}Cw_{t+1} \quad (21)$$

or

$$c_t = c_0 + \sum_{j=1}^t \hat{w}_j \quad \text{for} \quad \hat{w}_{t+1} := (1 - \beta)U(I - \beta A)^{-1}Cw_{t+1}$$

The unit root affecting c_t causes the time t variance of c_t to grow linearly with t .

In particular, since $\{\hat{w}_t\}$ is iid, we have

$$\text{Var}[c_t] = \text{Var}[c_0] + t \hat{\sigma}^2 \quad (22)$$

where

$$\hat{\sigma}^2 := (1 - \beta)^2 U(I - \beta A)^{-1} C C' (I - \beta A')^{-1} U'$$

When $\hat{\sigma} > 0$, $\{c_t\}$ has no asymptotic distribution.

Let's consider what this means for a cross-section of ex ante identical consumers born at time 0.

Let the distribution of c_0 represent the cross-section of initial consumption values.

Equation (22) tells us that the variance of c_t increases over time at a rate proportional to t .

A number of different studies have investigated this prediction and found some support for it (see, e.g., [3], [13]).

4.4 Impulse Response Functions

Impulse response functions measure responses to various impulses (i.e., temporary shocks).

The impulse response function of $\{c_t\}$ to the innovation $\{w_t\}$ is a box.

In particular, the response of c_{t+j} to a unit increase in the innovation w_{t+1} is $(1 - \beta)U(I - \beta A)^{-1}C$ for all $j \geq 1$.

4.5 Moving Average Representation

It's useful to express the innovation to the expected present value of the endowment process in terms of a moving average representation for income y_t .

The endowment process defined by (3) has the moving average representation

$$y_{t+1} = d(L)w_{t+1} \quad (23)$$

where

- $d(L) = \sum_{j=0}^{\infty} d_j L^j$ for some sequence d_j , where L is the lag operator Section ??
- at time t , the consumer has an information set Section ?? $w^t = [w_t, w_{t-1}, \dots]$

Notice that

$$y_{t+j} - \mathbb{E}_t[y_{t+j}] = d_0 w_{t+j} + d_1 w_{t+j-1} + \dots + d_{j-1} w_{t+1}$$

It follows that

$$\mathbb{E}_{t+1}[y_{t+j}] - \mathbb{E}_t[y_{t+j}] = d_{j-1} w_{t+1} \quad (24)$$

Using (24) in (16) gives

$$c_{t+1} - c_t = (1 - \beta)d(\beta)w_{t+1} \quad (25)$$

The object $d(\beta)$ is the **present value of the moving average coefficients** in the representation for the endowment process y_t .

5 Two Classic Examples

We illustrate some of the preceding ideas with two examples.

In both examples, the endowment follows the process $y_t = z_{1t} + z_{2t}$ where

$$\begin{bmatrix} z_{1t+1} \\ z_{2t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix}$$

Here

- w_{t+1} is an iid 2×1 process distributed as $N(0, I)$
- z_{1t} is a permanent component of y_t
- z_{2t} is a purely transitory component of y_t

5.1 Example 1

Assume as before that the consumer observes the state z_t at time t .

In view of (18) we have

$$c_{t+1} - c_t = \sigma_1 w_{1t+1} + (1 - \beta)\sigma_2 w_{2t+1} \quad (26)$$

Formula (26) shows how an increment $\sigma_1 w_{1t+1}$ to the permanent component of income z_{1t+1} leads to

- a permanent one-for-one increase in consumption and
- no increase in savings $-b_{t+1}$

But the purely transitory component of income $\sigma_2 w_{2t+1}$ leads to a permanent increment in consumption by a fraction $1 - \beta$ of transitory income.

The remaining fraction β is saved, leading to a permanent increment in $-b_{t+1}$.

Application of the formula for debt in (11) to this example shows that

$$b_{t+1} - b_t = -z_{2t} = -\sigma_2 w_{2t} \quad (27)$$

This confirms that none of $\sigma_1 w_{1t}$ is saved, while all of $\sigma_2 w_{2t}$ is saved.

The next figure illustrates these very different reactions to transitory and permanent income shocks using impulse-response functions

```
In [5]: const r = 0.05
const β = 1.0 / (1.0 + r)
const T2 = 20 # Time horizon
const S = 5 # Impulse date
const σ1 = 0.15
const σ2 = 0.15

function time_path(permanent = false)
    w1 = zeros(T2+1)
    w2 = similar(w1)
    b = similar(w1)
    c = similar(w1)

    if permanent === false
        w2[S+2] = 1.0
    else
        w1[S+2] = 1.0
    end

    for t=2:T2
        b[t+1] = b[t] - σ2 * w2[t]
        c[t+1] = c[t] + σ1 * w1[t+1] + (1 - β) * σ2 * w2[t+1]
    end

    return b, c
end

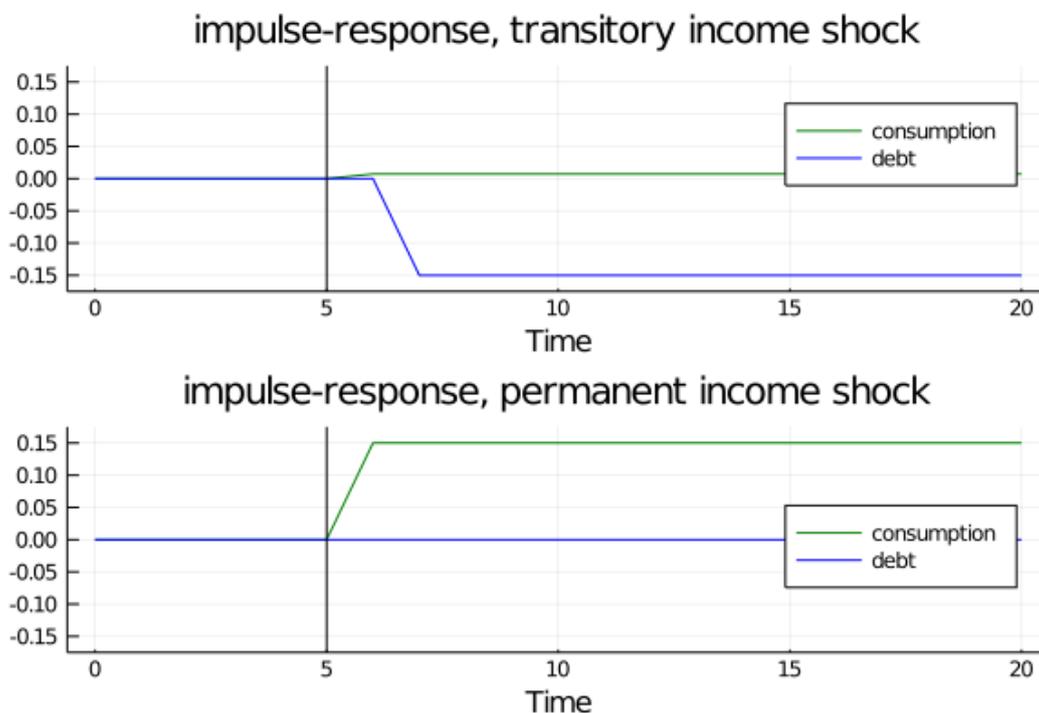
L = 0.175
```

```

b1, c1 = time_path(false)
b2, c2 = time_path(true)
p = plot(0:T2, [c1 c2 b1 b2], layout = (2, 1),
        color = [:green :green :blue :blue],
        label = ["consumption" "consumption" "debt" "debt"])
t = ["impulse-response, transitory income shock"
     "impulse-response, permanent income shock"]
plot!(title = reshape(t, 1, length(t)), xlabel = "Time", ylims = (-L, L),
      legend = [:topright :bottomright])
vline!([S S], color = :black, layout = (2, 1), label = "")

```

Out[5]:



5.2 Example 2

Assume now that at time t the consumer observes y_t , and its history up to t , but not z_t .

Under this assumption, it is appropriate to use an *innovation representation* to form A, C, U in (18).

The discussion in sections 2.9.1 and 2.11.3 of [11] shows that the pertinent state space representation for y_t is

$$\begin{bmatrix} y_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -(1-K) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ a_t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_{t+1}$$

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ a_t \end{bmatrix}$$

where

- K := the stationary Kalman gain
- $a_t := y_t - E[y_t | y_{t-1}, \dots, y_0]$

In the same discussion in [11] it is shown that $K \in [0, 1]$ and that K increases as σ_1/σ_2 does.

In other words, K increases as the ratio of the standard deviation of the permanent shock to that of the transitory shock increases.

Please see [first look at the Kalman filter](#).

Applying formulas (18) implies

$$c_{t+1} - c_t = [1 - \beta(1 - K)]a_{t+1} \quad (28)$$

where the endowment process can now be represented in terms of the univariate innovation to y_t as

$$y_{t+1} - y_t = a_{t+1} - (1 - K)a_t \quad (29)$$

Equation (29) indicates that the consumer regards

- fraction K of an innovation a_{t+1} to y_{t+1} as *permanent*
- fraction $1 - K$ as purely transitory

The consumer permanently increases his consumption by the full amount of his estimate of the permanent part of a_{t+1} , but by only $(1 - \beta)$ times his estimate of the purely transitory part of a_{t+1} .

Therefore, in total he permanently increments his consumption by a fraction $K + (1 - \beta)(1 - K) = 1 - \beta(1 - K)$ of a_{t+1} .

He saves the remaining fraction $\beta(1 - K)$.

According to equation (29), the first difference of income is a first-order moving average.

Equation (28) asserts that the first difference of consumption is iid.

Application of formula to this example shows that

$$b_{t+1} - b_t = (K - 1)a_t \quad (30)$$

This indicates how the fraction K of the innovation to y_t that is regarded as permanent influences the fraction of the innovation that is saved.

6 Further Reading

The model described above significantly changed how economists think about consumption.

While Hall's model does a remarkably good job as a first approximation to consumption data, it's widely believed that it doesn't capture important aspects of some consumption/savings data.

For example, liquidity constraints and precautionary savings appear to be present sometimes.

Further discussion can be found in, e.g., [7], [12], [2], [1].

7 Appendix: the Euler Equation

Where does the first order condition (5) come from?

Here we'll give a proof for the two period case, which is representative of the general argument.

The finite horizon equivalent of the no-Ponzi condition is that the agent cannot end her life in debt, so $b_2 = 0$.

From the budget constraint (2) we then have

$$c_0 = \frac{b_1}{1+r} - b_0 + y_0 \quad \text{and} \quad c_1 = y_1 - b_1$$

Here b_0 and y_0 are given constants.

Substituting these constraints into our two period objective $u(c_0) + \beta \mathbb{E}_0[u(c_1)]$ gives

$$\max_{b_1} \left\{ u \left(\frac{b_1}{R} - b_0 + y_0 \right) + \beta \mathbb{E}_0[u(y_1 - b_1)] \right\}$$

You will be able to verify that the first order condition is

$$u'(c_0) = \beta R \mathbb{E}_0[u'(c_1)]$$

Using $\beta R = 1$ gives (5) in the two period case.

The proof for the general case is similar.

Footnotes

[1] A linear marginal utility is essential for deriving (6) from (5). Suppose instead that we had imposed the following more standard assumptions on the utility function: $u'(c) > 0$, $u''(c) < 0$, $u'''(c) > 0$ and required that $c \geq 0$. The Euler equation remains (5). But the fact that $u''' < 0$ implies via Jensen's inequality that $\mathbb{E}_t[u'(c_{t+1})] > u'(\mathbb{E}_t[c_{t+1}])$. This inequality together with (5) implies that $\mathbb{E}_t[c_{t+1}] > c_t$ (consumption is said to be a 'submartingale'), so that consumption stochastically diverges to $+\infty$. The consumer's savings also diverge to $+\infty$.

[2] An optimal decision rule is a map from current state into current actions—in this case, consumption.

[3] Representation (3) implies that $d(L) = U(I - AL)^{-1}C$.

[4] This would be the case if, for example, the [spectral radius](#) of A is strictly less than one.

[5] A moving average representation for a process y_t is said to be **fundamental** if the linear space spanned by y^t is equal to the linear space spanned by w^t . A time-invariant innovations representation, attained via the Kalman filter, is by construction fundamental.

[6] See [8], [9], [10] for interesting applications of related ideas.

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